Stochastic Volatility and Option Pricing in the Brazilian Stock Market:
An Empirical Investigation

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Abstract

The stochastic volatility model proposed by Fouque et al. (SVFPS, 2000a) explores a rapid time-scale fluctuation of the volatility process to end up with a parsimonious way of capturing the volatility smile implied by close to the money options. In this paper, we test the SVFPS model using options from a Brazilian telecommunications stock. First, we find evidence of fast mean reversion in the volatility process. In addition, to test the model's ability to price options not so close to the money, we extend its statistical estimators to consider, in the calibration process, a wider region for the options moneyness. As an illustration, we price an exotic option.

Keywords: Option Prices, Stochastic Volatility, Mean Reversion, Mean Reversion Speed, Volatility Smile.
JEL Classification: C5,C51.

In a seminal paper, Black and Scholes (1973) introduced a pricing formula for options on an underlying stock following a geometric brownian motion. The assumptions include that the volatility of the underlying stock would be constant along the life of the option. From the empirical point of view, two different approaches show that this assumption is not valid. Regarding the historical time series of financial assets returns, many studies indicate that volatility is not constant along time[1]. Regarding the pricing measure, if we use market option prices on the same underlying with different maturities and exercise prices and invert the B&S formula to obtain implied volatilities, we observe that volatility is not constant. This phenomenon, denominated volatility smile, has been strongly evident on US. option market data, specially after the US stock market crash of 1987[2], and one can also observe it for options on Brazilian stocks. Figure 1 presents the volatilities of options with a fixed maturity of 24 workdays but with different moneyness (strike over actual stock price) on the underlying TNLP4.SA, a popular stock from a telecommunications firm[3]. The incompatibility of the constant volatility assumption with respect to market data motivated the study of more sophisticated models that could, consistently generate the smile pattern, as well as give a reasonable explanation for why such pattern arises.

Stochastic volatility models define an important class that extend the basic B&S model. Their main innovation is in considering the existence of an independent source of uncertainty driving the volatility of the underlying asset. Hull and White (1987) proposed one of the first published stochastic volatility models, which consisted of modelling the underlying stock price and the volatility process as Ito diffusions driven by independent Brownian Motions. After that, many models have been proposed, among them, the influential affine stochastic volatility model of Heston (1993), the stochastic volatility, stochastic interest rates and random return jumps model of Bakshi et al. (1997), and recently the double jump stochastic volatility model of Duffie et al. (2000) which allows for correlated jumps in the return and in the volatility processes.

All among previously cited models deal with data sampled on a daily basis. On the other hand, advances on the storage capacity of databases have recently allowed the development and test of econometric models using high-frequency data. On the high-frequency context, Fouque et al. (2000a) propose a fairly general formulation of a stochastic volatility model, the SVFPS model, which explores the existence of different time scales intrinsic in the dynamics of prices and volatilities, and provides corrections for B&S option prices that can be applied to price less liquid and exotic derivatives. Under fast mean reversion of the volatility process, the model is calibrated through a linear regression of implied volatility into the ratio log-moneyness / time to maturity. The only drawback of this model is that the approximation that implies this linear regression for the smile surface works well just for close to the money options[4].

In this paper, we test and calibrate the SVFPS model using option data from the Brazilian stock market. In general, financial institutions in emerging markets, are aware that illiquidity plays an important role on their daily trading game. In particular, illiquidity in the primary stock market directly propagates illiquidity to the option market, generating more pronounced smile patterns which are compatible with stochastic volatility models[5]. In the Brazilian equity market, although an acceptable level of liquidity exists in the primary market, it is the option market which is affected by the illiquidity phenomenon. For instance, even for the most liquid options, the moneyness pattern for a single stock is very sparse, usually spaced with 0.05 or 0.1 between options. In this case,

in order to calibrate any model to capture the dynamics of option prices, we need to make use of options with moneyness outside the range used in the previous applications of the SVFPS model. To this end, we propose different statistical estimators for the parameters of the model trying to accommodate the use of such not so close to the money options.

It should be clear that we have two different goals in this work. First, to identify the main qualitative characteristics of the volatility process which drives the dynamics of a typical stock in the Brazilian market. In this sense, we propose answering the following questions: Is the formulation of an ergodic mean reverting process compatible with this market? What is the speed of mean reversion for this process? What is the variance of the stationary distribution of the volatility process? The second goal, conditional on the market supporting evidence of a fast mean reverting stochastic volatility process, is related to how to calibrate the model using option data on the underlying stock including a region of the moneyness where the model has not been applied yet.

The paper is organized in the following sequence. In Section 1, we present the stochastic volatility model as long as the proposed extensions of the statistical estimators. In Section 2, we explain the variogram, the estimator of the speed of mean reversion of the volatility process. In Section 3, we present the empirical results of the paper, divided in two parts: The results related to the dynamics of the volatility and price processes, and the results on the calibration of the model using option prices on the previously cited Brazilian stock TNLP4.SA. Section 4 use the model to price an exotic binary option. We reserve Section 5 for final remarks and conclusion. The Appendix presents results on the sensitivity and statistical properties of the estimators of the parameters which appear in the dynamics of the stock price and stochastic volatility processes.

1   A Model for the Stock Price Dynamics

1.1   The Model

Consider the probability space (Ω, F, P) characterizing uncertainty. The following Stochastic Differential System describes the dynamics of the stock prices and its volatility:

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t, \tag{1}$$

$$\sigma_t = f(Y_t), \tag{2}$$

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t, \tag{3}$$

where W_t and $\hat{Z}_t$ are two Brownian Motions with correlation coefficient ρ, and f is a positive function, bounded above and away from zero. Process Y is a mean reverting Ornstein-Uhlenbeck process, which is well known to be a Gaussian process, with stationary distribution $N(m, \nu^2)$, where $\nu^2 = \frac{\beta^2}{2\alpha}$.

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In this model, volatility is an ergodic process that mean reverts to a long term mean which can be directly obtained from the long term mean m of process Y, once one defines an explicit form for function f[6]. In particular, the parameter α, the volatility mean reversion speed, plays an important role to determine the relative time scales which appear in the problem of modelling the dynamics of stock and option prices. If α lies in a region such that the volatility process presents a rapid time-scale flutuation relative to the time horizon of the option contracts, we say that the volatility is under a fast mean reversion regime.

This model presents some interesting characteristics. First, it produces leptokurtic and skewed density functions for the stock prices[7]. Second, it accounts for correlation between volatility and asset prices, a characteristic frequently identified in financial empirical studies[8]. Third, provided that the volatility mean reversion speed lies in the region that characterizes fast mean reversion, it presents very simple estimators based on the implied volatility surface to correct B&S prices to account for volatility randomness.

In the next two subsections we make the last point precise.

1.2   Option Pricing

This model characterizes an incomplete market, where the basic instruments (stocks and bonds) are not enough to span all the derivatives, specifically because volatility presents its on independent source of uncertainty. This extra uncertainty is charged, and its price γ_t is a process called the market price of volatility risk. In this context, to preclude arbitrages, we assume the existence of an equivalent martingale measure. The incompleteness of the market implies the existence of multiple equivalent martingale measures, each one parameterized by a particular process γ. Fixing one martingale measure Q_γ^*, let us consider pricing a derivative P on X, with payoff $H(X_T)$ at time T. The fact that the discounted price $\tilde{P}_t = \mathrm{e}^{-rt}P_t$ is a Q_γ^*-martingale guarantees that we can right:

$$P_t = E_\gamma^*[\mathrm{e}^{-r(T-t)}H(X_T) \mid \mathcal{F}_t] \tag{4}$$

An application of Girsanov's Theorem provides the dynamics of the asset price and volatility under the martingale measure Q_γ^*[9]:

$$dX_t = rX_tdt + \sigma_t X_t dW_t^*, \tag{5}$$

$$\sigma_t = f(Y_t), \tag{6}$$

$$dY_t = [\alpha(m - Y_t) - \beta(\rho\frac{(\mu - r)}{f(Y_t)} + \gamma_t\sqrt{1 - \rho^2})]dt + \beta d\hat{Z}_t^*, \tag{7}$$

$$\hat{Z}_t^* = \rho W_t^* + \sqrt{1 - \rho^2}Z_t^*. \tag{8}$$

Equation (4) is the Feynman-Kac representation for the solution of the following partial differential equation on P[10]:

$$\frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial y^2} + r(x\frac{\partial P}{\partial x} - P) + (\alpha(m-y) - \beta\Lambda(t,x,y))\frac{\partial P}{\partial y} = 0$$
(9)

where

$$\Lambda(t,x,y) = \rho\frac{(\mu - r)}{f(y)} + \gamma(t,x,y)\sqrt{1-\rho^2},$$
(10)

with terminal condition $P(T,x,y) = H(x)$.

1.3   Fast Mean Reversion and Asymptotic Analysis of Option Prices

Note that Equation (9) is the partial differential equation satisfied by the price of a derivative with payoff $H(X_T)$ at time T. If we are able to calculate the solution of this equation, then we now how to price derivatives under the proposed model. Under the fast mean reversion regime for volatility, the speed of mean reversion α is such that $\epsilon = \frac{1}{\alpha}$ represents a small quantity. This fact allows for the use of asymptotic theory of PDEs, to obtain an analytical approximation for the solution of the PDE (9), with an error in the approximation of order $\mathcal{O}(\epsilon)$. Moreover, this solution does not depend on the precise level of the volatility process, what guarantees that it is not necessary to estimate this process with precision[11]. This was basically the idea used in Fouque et al. (2000b). In what follows, we sketch the algorithm to obtain the approximate solution. Rewrite equation (9) using ϵ:

$$\frac{\partial P^\epsilon}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2 P^\epsilon}{\partial x^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\epsilon}}xf(y)\frac{\partial^2 P^\epsilon}{\partial x \partial y}$$
$$+\frac{1}{2}\frac{\nu^2}{\epsilon}\frac{\partial^2 P^\epsilon}{\partial y^2} + r(x\frac{\partial P^\epsilon}{\partial x} - P^\epsilon) + (\frac{1}{\epsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(t,x,y))\frac{\partial P^\epsilon}{\partial y} = 0$$
(11)

Then, expand the price function on a power series on $\sqrt{\epsilon}$:

$$P^\epsilon = P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3 + ...,$$
(12)

where $P_0, P_1, ...$ are functions of (t,x,y). After that, substitute Equation (12) into Equation (11), obtaining a geometric progression with ratio $\sqrt{\epsilon}$ with coefficients equal to different PDEs that must be solved[12]:

$$\frac{1}{\epsilon}L_0 P_0 + \frac{1}{\sqrt{\epsilon}}(L_0 P_1 + L_1 P_0) + (L_0 P_2 + L_1 P_1 + L_2 P_0) + \sqrt{\epsilon}(L_0 P_3 + L_1 P2 + L_2 P_1) + ... + = 0 \quad (13)$$

Truncate the series for all terms of order higher than $\sqrt{\epsilon}$ and solve the four remaining PDEs to obtain the following $\mathcal{O}(\epsilon)$ approximation for the price of a derivative P with strike price k, at time t:

$$P(X_t, k, \bar{\sigma}) = P_0(X_t, k, \bar{\sigma}) + P_1(X_t, k, \bar{\sigma}) + \mathcal{O}(\epsilon),$$
(14)

where P_0 is the B&S price of the derivative with constant volatility equal to $\bar{\sigma}$[13], which satisfies $P_0(X_T, k, \bar{\sigma}) = H(X_T)$. For instance, in the case of a call option, it is given by:

$$
\begin{aligned}
P_0(X_t, k, \bar{\sigma}) &= X_t N(d_1) - k e^{-r(T-t)} N(d_2), \\
d_1 &= \frac{log(X_t/k) + (r + \frac{1}{2}\bar{\sigma}^2)(T-t)}{\bar{\sigma}\sqrt{T-t}} \\
d_2 &= d_1 - \bar{\sigma}\sqrt{T-t} \\
N(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2}} dy
\end{aligned}
\tag{15}
$$

P_1 is a model correction term that satisfies $P_1(X_T, k, \bar{\sigma}) = 0$, and is given by:

$$
P_1(X_t, k, \bar{\sigma}) = -(T-t)(V_2 X_t^2 \frac{\partial^2 P_0(X_t, k, \bar{\sigma})}{\partial x^2} + V_3 X_t^3 \frac{\partial^3 P_0(X_t, k, \bar{\sigma})}{\partial x^3}),
\tag{16}
$$

where the partial derivatives are taken with respect to the first argument of P_0. The parameters V_2 and V_3 are obtained along the asymptotic analysis and have completely different interpretations. V_2 is related to the market price of volatility risk and represents a volatility level correction that incorporates the fact that volatility is random. On its turn, V_3 is related to the third moment of the stock, and vanishes when $\rho = 0$. So a natural interpretation for V_3 is that it is a parameter directly related to the skewness of the returns distribution. Fouque et al. (2000b) show that at this order of approximation for the prices, the "smile effect" will only be reproduced in the correlated case. Even in the case that ρ is not zero, it may happen that you will not be able to capture the particular smile you are interested in analyzing, using this model. For instance, if ρ is negative, then the smile[14] will be downward sloping. This is reasonable, since the negative ρ contributes to generate a fat left tail increasing prices of in the money and close to the money calls (out of the money and close to the money puts), and a thinner right tail decreasing prices of out of the money calls. When the option data being analyzed is composed by points that are near the money, the effect of the extremes is not considered and this approximation for the smile sounds reasonable. On the other hand, when we also consider out of the money data, we are not able to perfectly reproduce a picture like Figure 1, as will be shown in the empirical section.

In their work, they also propose an ϵ expansion for the implied volatility function obtaining a very nice linear relation between implied volatility and the ratio log moneyness to time to maturity:

$$
I = a \left[\frac{log\left(\frac{\text{strike price}}{\text{asset price}}\right)}{\text{time to maturity}} \right] + b + \mathcal{O}\left(\frac{1}{\alpha}\right),
$$

$$
a = -\frac{V_3}{\bar{\sigma}^3},
\tag{17}
$$

$$
b = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left(r + \frac{3}{2}\bar{\sigma}^2\right) - \frac{V_2}{\bar{\sigma}}.
$$

It is demonstrated in Fouque et al. (2000b) that the last equation works as a good approximation, when the region of moneyness that is being studied is very close to the money[15]. However, for applications in Emerging Markets, this point should be reviewed

because the liquidity is much more restrictive, with sequential strike prices of options on the same asset, usually jumping at least 5% in the moneyness. This approximations relies on the fact that for close to the money options, the implied volatility can be approximated as a linear function of the option prices. Figure 6 shows the implied volatility as a function of option prices[16]. Note how the out of the money call options introduce a non linearity in the relation Implied Volatility versus Option Prices. For this reason, we decide to provide additional estimators that avoid expanding the implied volatility and truncating its series. Using these estimators, we expect to obtain the parameters V_2 and V_3 providing a better fit of the model to the option market prices[17]. We relied on Equation (14) to fit the option market prices, choosing V_2 and V_3 to minimize the following loss function:

$$\sum_{i=0}^{K} W_i (Price_i - P_t(X_t, k_i, \bar{\sigma}))^2, \tag{18}$$

where $Price_i$ represents the market price of the ith option, P_t is the proposed model price for the ith option, k_i represents the strike price of the ith option, and W_i represents a weight value for the ith term in the sum. Note that if we use all weights equal to one, we obtain the Ordinary Least Squares Estimator, that minimizes the sum of the absolute square residuals in the prices. In addition, note that in this case we can implement the estimator by making use of a linear multiple regression of $Price$ on V_2 and V_3. The problem with this approach relies on the fact that it gives too much weight to the options that are in the money and close to the money, comparing to the options that are out of the money, because the prices of out of the money options are much smaller than their counterparts. For this reason, we also experienced with weight i being equal to $\frac{1}{Price_i^2}$, that minimizes the sum of the squares of the relative error as a percentage of price. In this case, the out of the money options play a very important role in the choice of the parameters. However, the price that this estimator pays is that it provides big residuals for the close to the money and in the money options. In subsection 3.2, we test the three estimators presented here.

2 The Variogram

This section introduces the variogram as a tool for the estimation of the mean reversion rate α.

Let us first consider a discrete version of time, with N units of Δt size[18]. Adopting a discrete version of Equation (1), we define the demeaned normalized fluctuations of data by:

$$D_n = \frac{1}{\sqrt{\Delta t}} \frac{\Delta X_n}{X_n} - \mu \sqrt{\Delta t} = f(Y_n) \frac{\Delta W_n}{\sqrt{\Delta t}}, n = 1, ..., N \tag{19}$$

where X_n, Y_n and W_n represent the processes X_t, Y_t and W_t sampled at time $n\Delta t$. From the basic properties of Brownian Motion we are able to rewrite D_n as:

$$D_n = f(Y_n)\epsilon_n, n = 1, ...N \tag{20}$$

where ϵ_n is a sequence of IID Gaussian random variables with mean zero and variance 1. We further define, in order to obtain additive noise process, the log absolute value of the

normalized fluctuations:

$$L_n = log(|D_n|) = log(f(Y_n)) + log(|\epsilon_n|) \tag{21}$$

The variogram is a measure of the correlation structure of the sampled version of the process $log(f(Y_t))$:

$$V_j^N = \frac{1}{N} \sum_{n=1}^{N} (L_{n+j} - L_n)^2, j = 1, 2, ..., J; \tag{22}$$

where j is the lag for which you are measuring the correlation.

Fouque et al. (2000b) show that for each $j = 1, 2, ..., J$, V_j^N is an unbiased estimator of $2c^2 + 2\nu_f^2(1 - e^{-j\alpha\Delta t})$, where $c^2 =$ variance($log|\epsilon|$), and ν_f^2 is the variance of the stationary distribution of the process $log(f(Y_t))$. Using this result, we are able to estimate by standard non-linear regression methods, the volatility mean reversion speed α. We use the empirical variogram from Equation (22) as the dependent variable, and adjust it to the functional form $2c^2 + 2\nu_f^2(1 - e^{-j\alpha\Delta t})$, optimizing on the parameters c, ν_f and α.

3 Brazilian Equity Market: Empirical Results

We divide the empirical study of this paper in two parts: The results related to the dynamics of the volatility and price processes of a stock in an Emerging Market, and the results regarding the calibration of the model using option prices on this stock.

3.1 Analysis of the Returns, Variogram, and Estimation of α

In a previous study, Fouque et al. (2003) identify a fast mean reversion regime for the volatility of the returns of the S&P 500 index. We believe, it is of general interest to study the qualitative properties of the volatility process that drives the prices of financial securities. In particular, our goal in this subsection is to verify if the volatility process driving a typical stock in an Emerging Market presents the same behavior as the volatility process driving the S&P 500 index, an important instrument representing the stock market of a developed country. We use historical data on TNLP4, a stock in the telecommunications market, to answer the following interesting questions: Is the volatility a mean reverting process? Is the speed of mean reversion comparable to the speed of mean reversion of the S&P 500 volatility? Is the variance of the stationary distribution of the volatility higher than the correspondent in the US. market? We believe this part of the study deserves its own attention, because it may help researchers to better understand the qualitative time series properties of the volatility process of a financial stock in an Emerging market.

Figure 2 depicts the normalized TNLP4 returns, using a 30 minutes sampling rate[19]. Observe that the amplitude of the returns frequently changes along time presenting the same pattern as the S&P500 returns reported in Fouque et al (2003). This may be seen as a first evidence in favor of a fast mean reversion regime for the volatility of TNLP4

returns. In order to provide more intuition, we simulate prices from the discrete stochastic volatility model presented in Section 2, using $f(y) = e^y$, assuming different speeds of mean reversion, but maintaining constant the variance ν^2 of the invariant distribution of the Ornstein Uhlenbeck process. The upper portion of Figure 3 depicts the correspondent volatility processes for a mean reversion speed equal to 1 and $\frac{1}{200}$ 1/days respectively. The lower portion presents the correspondent return processes. Compare Figure 2 to Figure 3 to observe the similarity between the TNLP4 returns and the return process generated by the volatility process with high mean reversion speed.

At this point, based on the equations described in Section 2, we use the Variogram to obtain an estimation of the volatility mean reversion speed α. Figure 4 shows the empirical and the fitted variograms. Table 1 presents the estimated values and standard deviations for the parameters of interest, α and ν. Standard deviations were obtained using a simple non-parametric bootstrap on the residuals of the non linear regression solved to obtain the Variogram, using a total of 10000 bootstrap samples. The estimated volatility mean reversion speed is 0.476 $\frac{1}{\text{days}}$, with standard deviation equal to 0.0432, meaning an average decoupling time of 2.1 days, with a 95% confidence interval $[1.7764, 2.4236]$. The low standard deviation indicates that if we were to design a hypothesis test to verify if the volatility process is a mean reverting process:

$$\begin{aligned} H_0 &: \alpha = 0 \\ H_1 &: \alpha > 0 \end{aligned} \tag{23}$$

at a 99% confidence interval, we would reject the null hypothesis, confirming the volatility process as a mean reverting process. Moreover, the value 0.476 indicates that the volatility process of the TELEMAR stock is under the regime of fast mean reversion.[20] The volatility ν of the invariant distribution of the Ornstein-Uhlenbeck process Y was estimated to be 0.32. In Figure 5 we present the empirical and fitted variogram for a simulated path of the price process, using the discrete model described in Section 2, with the parameters estimated in the present section. Now compare Figure 4 to Figure 5 to note the similarity of the variograms obtained with real and simulated data[21]. This is a good indication that we are being able to capture the empirical properties of the historical stock prices with our model approach.

Comparing the result obtained for the speed of mean reversion of the OU process driving the volatility of the Brazilian stock, $\alpha_{Brazil} = 0.476$ with the result obtained for the S&P500, $\alpha_{US} = 0.667$, we conclude that both markets have a similar pattern with respect to the speed of mean reversion, that is a fast mean reversion regime for the volatility. On the other hand, in order to extract information from the variance of the stationary distribution of the OU process driving the volatilities in both markets, $\nu_{Brazil} = 0.32$ and $\nu_{US} = 0.26$, we first need to estimate the effective volatility $\bar{\sigma}^2$. We estimated the effective volatility, calculating the annualized standard deviation of the normalized fluctuations process presented in Figure 2, obtaining $\bar{\sigma}^2_{Brazil} = 43.2\%$[22]. The value obtained in Fouque et al (2003) for the S&P500 index was $\bar{\sigma}^2_{US} = 0.07$[23]. The effective volatility is a proxy for the variability of the volatility process. So if we compare the effective volatilities, we see that the variability of the volatility in the Brazilian market is much higher than the variability of the volatility in the US market. Going deeper in the model, using the additional result that $\bar{\sigma}^2 = e^{2m+2\nu^2}$, and the estimated effective volatilities, we conclude that

the long term mean m of the OU driving the volatility process is significantly higher in the Brazilian market, although the variance of the stationary distribution of the OU are of the same order in both markets.

3.2   Stochastic Volatility, Option Prices and the Correction Term

Table 2 presents prices and volume for calls on the TELEMAR stock on November 13, 2002, at 13:30. In order to fit the model to the implied volatility smile generated by the data in Table 2, we implemented the three estimators described in subsection 1.3 for the parameters V_2 and V_3: the Implied Volatility estimator (IV), the Absolute Price estimator (OLS), and the Weighted Relative Price estimator (WRP). First, we made use of the complete sample of option prices, including the three out-of-the-money options (OM). The estimated values of V_2 and V_3, and their respective standard deviations appear in Table 3. Figure 7 shows the fit of each estimator to the implied volatility smile. The implied volatilities for IV estimator were obtained using Equation (17), while for OLS Price Estimator and WRP Estimator, were obtained calculating the implied volatilities inverting $P_0 + P_1$ which appear in Equation (14). The standard deviations for the IV estimator where obtained directly from the linear regression suggested in Equation (17). Similarly, for the OLS price estimator, the standard deviations of the parameters were obtained directly from the linear regression suggested in Equation (14). The WRP estimator is based on Generalized Least Squares method. In this case, we relied on a bootstrap scheme for regressions. We first applied a simple non-parametric bootstrap scheme, with 10000 resampling size (see Davidson and Mackinnon (1993)). As the standard deviations were so large, and also because we can not guarantee that the relative price errors are homoscedastic, we also experimented with a parametric Heteroscedastic Gaussian bootstrap (see Cribari and Zarkos (1999)). Although not reported, the results obtained were very similar.

Table 3 shows that the parameters for the WRP estimator have no significance[24]. On the other hand, the parameters for the other two estimators are clearly significant. Observe also that for the first two estimators, the signs and the order of the estimated parameters are the same. Actually their values for V_2 and V_3 are similar to the results in Fouque et al. (2003). On the other hand, the WRP estimator presents a much smaller V_2, and V_3 with an opposite sign. From the fitting point of view, none of the estimators work well for the complete option data set. Here we outline some possible reasons for that. Noticing that all the options have a common maturity, and that $log(1 + x) \approx x$ for small x, Equation (17) shows that the IV estimator fits a line to the implied volatilities curve as a function of moneyness. Figure 1 shows that the implied volatility is clearly not a linear function of moneyness. OLS Price estimator gives too much weight to the in-the-money options, which worth much more than the out of the money options. The result is a better fit to the in-the-money options and a deteriorate fit to the out-of-the-money options, when compared to IV estimator. Last, WRP estimator gives too much weight to the out-of-the-money options, and as a result one can see that it fits very well the implied volatilities of the OM options but presents very poor results for the in and close-to-the-money options, with a smile inverted[25]. We present in Table 4 the square root of the mean square error

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($SRMSE$) to compare the fitting of the implied volatility surface among the different estimators. The $SRMSE^j$ measures the average error implied by model j in fitting all the implied volatilities. Among the three estimators the one that achieves the smallest $SRMSE$ is the IV estimator. Although the new estimators do not present a good performance with respect to the fitting of the implied volatility curve, the WRP estimator presents a very reasonable result when the performance measure is shifted to the relative price error $\frac{|P_{observed}-(P_0+P_1)|}{P_{observed}}$. Figure 8 shows the relative price errors for the three estimators. The relative errors for the WRP estimator are all less than 10% while errors for the other two estimators blow up for OM options.

Searching for a region of the data where the model presents a fair fit, we adjust the estimators using only in and close-to-the-money options data, leaving aside the three OM observations. From Table 2 we can see that although we discard information on the call option with strike 28, which presents a considerable volume, we include the options with strike 24 and 26 which present the highest volume of that day. Table 5 presents the parameters values and standard deviations, and Table 6 presents $SRMSE$ for the different estimators. The first interesting thing to note is that the values of V_2 and V_3 are very similar for the three estimators. This indicates that the region of OM data is the main responsible for the distinction between the estimators. We can also see from Table 5 that all the parameters are significant, and that in particular WRP estimator that presented the highest standard deviations on Table 3, presents the smallest now. Table 6 shows that the IV estimator has a very good fit with an average error of 0.6% in the implied volatility. Figure 9 shows the fit of each estimator to the implied volatility smile.

4 Application: Pricing an Exotic Derivative

In this Section, we present an application of the model for pricing an exotic option, using the calibration obtained in 3.2 for in and close to the money options.

4.1 Pricing an European Binary Option

Denote by X_t the stock price at time t. An European binary call with strike price k, and maturity T, pays a fixed amount U on date T, if $X_T > k$, and pays zero if $X_T \leq K$. Its payoff function is:

$$H(X_T) = U1_{\{X_T > k\}} \tag{24}$$

where 1 denotes the indicator function.
According to the results outlined in Section 1.2, the price of this option is obtained by the following expectation under the risk neutral measure:

$$P_t = Ue^{-r(T-t)}E_\gamma^*[1_{\{X_T>k\}} \mid \mathcal{F}_t] = Ue^{-r(T-t)}Q_\gamma^*[X_T > k \mid X_t] \tag{25}$$

The Black & Scholes price of this option is obtained by using the fact that, under the martingale measure in the B&S model, X_t follows a geometric Brownian Motion with drift r and constant diffusion coefficient $\bar{\sigma}$. This implies that $\log X_T$ under B&S follows a

normal distribution with mean $\log X_t + (r - \frac{1}{2}\bar{\sigma})(T-t)$ and variance $\bar{\sigma}^2(T-t)$. With this information we are able to calculate:

$$Q^*_{B\&S}[X_T > k \mid X_t] = Q^*_{B\&S}[\log X_T > \log k \mid X_t] = N(d_2), \qquad (26)$$

where $N(d_2)$ is defined in Equations (15).
Using Equation (26) we obtain the B&S price of the binary call:

$$P_0(X_t, k, \bar{\sigma}) = U\mathrm{e}^{-r(T-t)}N(d_2) \qquad (27)$$

Note that d_2 depends on k. This is the only factor that contributes for different B&S binary prices according to different strikes.
On the other hand, according to the stochastic volatility model, an approximation for the price of such binary option should be given by:

$$P(X_t, k, \bar{\sigma}) = P_0(X_t, k, \bar{\sigma}) + P_1(X_t, k, \bar{\sigma}) = U\mathrm{e}^{-r(T-t)}N(d_2) + P_1(X_t, k, \bar{\sigma}), \qquad (28)$$

where

$$P_1(X_t, k, \bar{\sigma}) = -(T-t)(V_2 X_t^2 \frac{\partial^2 P_0(X_t, k, \bar{\sigma})}{\partial x^2} + V_3 X_t^3 \frac{\partial^3 P_0(X_t, k, \bar{\sigma})}{\partial x^3}), \qquad (29)$$

Calculating the partial derivatives which appear in Equation (29) using Equation (27), Fouque et al. (2000b) obtained:

$$P_1(X_t, k, \bar{\sigma}) = U \frac{(\frac{X_t}{k})\mathrm{e}^{-\frac{d_1^2}{2}}}{\bar{\sigma}^2\sqrt{2\pi}} \left(\frac{V_3}{\bar{\sigma}\sqrt{T-t}}(1 - d_1^2) - (V_3 - V_2)d_1 \right), \qquad (30)$$

where d_1 is defined in Equation (15).

The values of the variables of interest observed on November, 13, 2002 in the Brazilian market were $X_t = 23.6$, $r = 0.23$. Combining this information with the estimated $\bar{\sigma} = 0.432$, we were able to obtain the prices for binary calls on TELEMAR stock for all the different strikes considered in the estimation process. Figure 12 plot both the B&S prices (solid line) and the model implied prices (dashed line) for $U = 1$. Note how different these prices can be. Figure 13 plots the ratio between the correction term P_1 and the total price given in Equation (28). Note that for options with strikes 24 and 26, the correction is greater than 10% of the model implied price, indicating that for a market maker, for instance an investment bank, a robust model that accounts for volatility randomness should be considered in order to price and hedge exotic options.

5 Conclusion

In this paper, we present an empirical study of the Brazilian equity market, adopting the stochastic volatility model proposed in Fouque et al. (2000a). From the theoretical viewpoint, the model is consistently built under the assumptions of no-arbitrage. From the practical viewpoint, it is parsimonious and not difficult to calibrate. In the first part of our empirical study, in order to be able to compare the qualitative characteristics of the volatility processes of a Brazilian stock and of the S&P 500 index, we estimated the

volatility mean reversion speed, the variance of the stationary distribution of the OU process driving the volatility process, and the effective volatility of the Brazilian TELEMAR stock. Comparing these results with the S&P 500 results reported in Fouque et al. (2003), we conclude that both markets are under a fast mean reversion regime, with speed of mean reversion with the same order of magnitude, and that the variance of the volatility process is considerably higher in the Brazilian market. This higher level for the Brazilian volatility represents no surprise, in the sense that it is usually attributed to uncertainty regarding economical and political events. On the other hand, the fast mean reversion speed for the Brazilian volatility process provides room for the application of the SVFPS model.

The fast mean reversion regime indicates that the model approximations for derivatives pricing work well for close to the money options. However, in Emerging Markets, even the close to the money options are usually 5% to 20% far from the at the money strike. For this reason, when calibrating the model, we experimented with two different estimators also implied from the model but not previously proposed, namely, the OLS Price estimator, and the WRP estimator. The first gives uniform weight for the model implied errors in the prices of options with different strikes, while the second gives more weight for the errors in the prices of out of the money options. The empirical results showed that none of the estimators is satisfactory when we include out of the money options in the calibration of the implied volatility curve, though the WRP estimator had a reasonable performance in terms of relative price errors (all less then 10%). On the other hand, when we consider close to the money options[26] (moneyness from 0.85 to 1.10), the IV estimator works reasonable well with an average error of 0.6% in the implied volatility fitting. We then use the model implied parameters to price an exotic European call giving an example of how far the prices can be from the correct ones, when one assumes B&S model for pricing.

Although the two new estimators do not perform well, this paper gives one possible direction for further research on the search of new estimators[27] which will allow the application of the SVFPS model in a variety of markets, in contrast to extremely liquid markets like the S&P 500 market.

6   Acknowledgements

References

[1] Ait Sahalia Y., and A. Lo, 1998, "Non Parametric Estimation of State-Price Densities Implicit in Financial Asset Prices," *Journal of Finance*, 53, 499-547.

[2] Bakshi G., C. Cao, and Z. Chen, 1997, "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52, (5), 2003-2049.

[3] Bakshi G., C. Cao, 2002, "Risk-Neutral kurtosis, Jumps, and Option Pricing: Evidence from 100 Most Actively Traded Firms in CBOE," working Paper, University of Maryland.

[4] Black F. and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.

[5] Barndorff-Nielsen O.E. and N. Shephard, 2002, "Econometric Analysis of Realised Volatility and Its Use in Estimating Stochastic Volatility Models," *Journal of the Royal Statistical Society, Series B*, 64, 253-280.

[6] Brandt M. and P. Santa Clara, 2001, "Simulated Likelihood Estimation of Diffusions with Application to Exchange Rate Dynamics in Imcomplete Markets," *Journal of Financial Economics*, 63, 2002, 161-210.

[7] Cribari-Neto F. and S. Zarkos, 1999, "Bootstrap Methods for Heteroskedastic Regression Models: Evidence on Estimation and Testing," *Econometric Reviews*, 18, 211-228.

[8] Davidson R., and J.G. Mackinnon, 1993, Estimation and Inference in Econometrics, Oxford University Press, NY.

[9] Duffie D., 2001, Dynamic Asset Pricing Theory (third edition), Princeton University Press, NJ.

[10] Duffie D., J. Pan and K. Singleton, 2000, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343-1376.

[11] Dumas B., J. Flemming, and R. Whaley, 1998, "Implied Volatility Smiles: Empirical Tests," *Journal of Finance*, 53, 6, 2059-2106.

[12] Hull J. and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance* 42, (2), 281-300.

[13] Fouque J.P, G. Papanicolaou, and R. Sircar, 2000a, "Mean Reverting Stochastic Volatility," *International Journal of Theoretical and Applied Finance*, 3, (1), 101-142.

[14] Fouque J.P, G. Papanicolaou, and R. Sircar, 2000b, Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press, NY.

[15] Fouque J.P, G. Papanicolaou, R. Sircar, and K. Solna, 2003, "Short Time Scale in S&P500 Volatility," *Journal of Computational Finance*, 6, 4, 1-23.

[16] Gallant R., and G. Tauchen, 1997, "Estimation of Stochastic Volatility Models with Diagnostics," *Journal of Econometrics*, 81, 159-192.

[17] Heston S., 1993, "A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, (2), 327-343.

[18] Jun S-G, A. Marathe and H.A. Shawky, 2003, "Liquidity and Stock Returns in Emerging Equity Markets," *Emerging Markets Review*, 4, 1, 1-24.

[19] Kim S., N. Shephard, and S. Chib, 1998, "Stochastic Volatility: Likelihood Inference and Comparison with Arch Models." *Review of Economic Studies*, 65, 361-393.

[20] Pedersen A., 1995, "A New Approach for Maximum Likelihood Estimation of Stochastic Differential Equations Based on Discrete Observations," *Scandinavian Journal of Statistics*, 22, 55-71.

7 Appendix: Sensitivity and Statistical Properties of the Estimators of α, ν, and $\bar{\sigma}$

In this Appendix, assuming that $f(y) = e^{y}$[28], we derive some analytical expressions, some statistical properties, and provide some simulation results for the estimators of α, ν, and $\bar{\sigma}$.

7.1 Properties of the Estimator of $\bar{\sigma}$

In this Appendix, we give evidence for why $D_N = \sum_{n=1}^{N} \frac{D_n^2}{N}$ should be used as an estimator for $\bar{\sigma}^2$. First, we give conditions under which this estimator would be unbiased and asymptotically consistent. Later, identifying that these conditions are not usually satisfied, we provide empirical computational evidence based on monte carlo simulations.

By definition, according to Section 2, D_n is composed by the product of $f(Y_n)$ and ϵ_n, where ϵ_n is $N(0,1)$[29]. The two discrete sequences ϵ_n and Y_n present correlation coefficient ρ, which represents the correlation coefficient of the two Brownian Motions introduced in the model on Section 1. Suppose, as a simplifying condition, that the correlation coefficient is zero. Then, when we calculate the second moment of D_n, $E[D_n^2]$, the independence of the Brownian Motions allows the separation of the expectation into the product $E[f(Y_n)^2]E[\epsilon_n^2]$, which is clearly equal to $E[f(Y_n)^2]$. So, when ρ equals zero, we obtain the following expressions for the mean and the variance of D_N[30]:

$$E[D_N] = \sum_{n=1}^{N} \frac{E[D_n^2]}{N} = E[f(Y_n)^2] = \bar{\sigma}^2. \tag{31}$$

$$Var[D_N] = E[D_N^2] - \bar{\sigma}^4 = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{E[f(Y_i)^2 f(Y_j)^2]E[\epsilon_i^2 \epsilon_j^2]}{N^2} - \bar{\sigma}^4. \tag{32}$$

16

At this point, reminding that the invariant distribution of the Ornstein-Uhlenbeck process Y is $N(m, \nu^2)$, and using the log-normal property that $E[e^Y] = e^{m+\frac{\nu^2}{2}}$, we obtain, when $i = j$, $E[f(Y_i)^2 f(Y_j)^2] = E[e^{4Y}] = e^{4m+\frac{16\nu^2}{2}}$. For $i \neq j$, we use the following fact from Fouque et al. (2000a): $E[f(Y_i)^2 f(Y_j)^2] = \bar{\sigma}^4 e^{4\nu^2 e^{-\alpha|j-i|\Delta t}}$. Making the same analysis for the ϵ_s terms, we obtain, for $i = j$, $E[\epsilon_i^2 \epsilon_j^2] = E[\epsilon^4] = 3$, otherwise, by the uncorrelated structure of the ϵ_s series, $E[\epsilon_i^2 \epsilon_j^2] = 1$. Combining all this information, and separating terms in the summation which appears in Equation (32), we obtain:

$$Var[D_N] = \bar{\sigma}^4 \left[\frac{3e^{4\nu^2}}{N} + \frac{2}{N^2} \sum_{n=1}^{N-1} (N-n) e^{4\nu^2 e^{-\alpha n \Delta t}} \right] - \bar{\sigma}^4. \tag{33}$$

The previous calculation shows that the estimator is unbiased, and a simple computation using typical values for α, ν and Δ_t indicates that the estimator is also consistent, with asymptotic volatility converging to zero. Figure 11 shows this convergence with parameters values equal to the ones obtained in the empirical section. However, the correlation coefficient is not zero, and according to the empirical literature, it is usually negative. Making use of the discrete version of the model proposed in Section 1, and assuming $f(Y) = e^Y$, we realized monte carlo simulations with the set of model parameters fixed, except for the correlation coefficient ρ, intending to show that our estimator for $\bar{\sigma}$ is not sensitive to variations on the parameter ρ. For each value of ρ in the set $\{-0.9, -0.8, -0.7, -0.5, -0.3, -0.1, 0.0\}$, we simulated 500 sample paths for the returns and volatilities, each path with 4291 elements, the size of the TNLP4 series available in our empirical study. For each of these 500 samples, we calculate the expectation of the squared volatility process, calculate the expectation of the squared D_n series, and obtain the relative error as the difference of these two quantities divided by the first, which is the one we are looking for. In Figure 10, we plot the histograms of the relative error, for all values of ρ. One can note that the performance of the estimator is very good and independent of ρ, justifying our estimation procedure of using the second moment of D_n to estimate $\bar{\sigma}$.

7.2 Testing the Estimators of α and ν with Respect to the Correlation Coefficient ρ

The expression for the Variogram provided in Section 2 and obtained in Fouque et al. (2000a) is valid under the hypothesis that the correlation coefficient ρ is zero, or in other words, that the Brownian Motions driving the asset prices and the volatility dynamics are independent. In justifying the use of such expression to estimate the parameters α and ν using real data, where there exists non-zero correlation between the Brownian Motions, Fouque et al. (2000a) provide monte carlo simulations results showing that the sensitivity of the estimator to different values of ρ is very small. In this Appendix, we reinforce their results providing our own monte carlo experiment. For each value of ρ in the set $\{-0.9, -0.8, -0.7, -0.5, -0.3, -0.1, 0.0\}$, using $\frac{1}{\alpha} = 2.1$ days and $\nu = 0.32$, we simulated 500 sample paths for the returns and volatilities, each path with 4291 elements, the size of the TNLP4 series available in our empirical study. For each sample, we use the Variogram method to estimate the parameters. We present on Figure 14 the

sensitivity of the estimators of α and ν with respect to ρ. Table 7 presents the mean and standard deviation of the estimators, calculated using the 500 estimated values from each different correlation coefficient ρ. Note that although the average values for $\frac{1}{\alpha}$ are slightly underestimated and that the average values for ν are slightly overestimated, these patterns persist for all different values of ρ, implying just minor differences among the results.

Notes

[1]For instance see Kim et al. (1998), Barndorff-Nielsen and Shephard (2002).

[2]According to Ait-Sahalia and Lo (1998) and Dumas et al. (1998), among others.

[3]The implied volatilities where obtained using the TNLP4 price at 13:30, which was of R$ 23.6, and annualized interest rates equal to 23%. Interest rates where obtained from the term structure of the swaps floating-fixed, available in the BM&F database (www.bmf.com.br).

[4]All the results reported in Fouque et al. (2000a) (2000b) (2003) are for options with moneyness in between 0.97 and 1.03.

[5]For a study on the effect of liquidity in the returns of emerging market equities see Jun et al. (1999).

[6]A more general version of the model only assumes that Y is an ergodic Markov Ito process, and is described in Fouque et al. (2000b).

[7]The randomness of volatility contributes for the excess kurtosis, while the correlation between the Brownian Motions produces asymmetry.

[8] Actually, there is strong evidence indicating that the correlation coefficient between volatility and asset prices is negative. See Bakshi et al. (1997), Bakshi et al. (2002) and Heston (1993).

[9]See Fouque et al. (2000b), page 47.

[10]See Duffie (2001).

[11]Estimation of stochastic volatility processes that are latent processes, demands the use of computational intensive methods, which are very hard to calibrate, being among them, Simulated Method of Moments (Gallant and Tauchen (1997)), Simulated Maximum Likelihood Estimator (Pedersen (1995), Brandt and Santa Clara (2002)), or Monte Carlo Markov Chain Methods (Kim et al. (1998)).

[12]$L_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}$, $L_1 = \sqrt{2}\rho\nu x f(y)\frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu\Lambda(y)\frac{\partial}{\partial y}$, $L_2 = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - .)$.

[13]$\bar{\sigma}$, denominated effective volatility, is defined as the square root of the expectation of the variance process $f(Y)^2$, with respect to the invariant measure: $\bar{\sigma} = \sqrt{E[f(Y)^2]}$, where Y is $N(m, \nu^2)$. Fouque et al. (2000b) have shown that in a fast mean reversion regime for volatility, $\bar{\sigma}^2$ can be thought as a first approximation for the process $f(Y)^2$, although, except for the extremely fast mean reversion case, usually one or two extra volatility terms are necessary for a good approximation.

[14]The implied volatility as a function of exercise prices.

[15]In their S&P500 study they use option data with moneyness in between 0.97 and 1.03.

[16]It uses the same data that generated Figure 1.

[17]It is important to remark that this argument was not theoretically demonstrated.

[18]In our empirical study we use Δt=30 minutes.

[19]The returns were normalized from 30 minutes returns to daily returns.

[20]We adopt the qualitative convention that a decoupling time of less than a week characterizes the fast mean reversion regime. Quantitative tests regarding the accuracy of the asymptotic approximation might be realized in a posterior step, when using option data.

[21]The only difference is a translation on the y-axis for the position where the variogram is located. This translation is related to the difficult of estimating the variance of the logarithm of the absolute value of the ϵ term that appears in Section 2. However, this translation is not important because we are just interested

in the curvature and height of the variogram for practical purposes.

[22]In the Appendix we show that $\sum_{n=1}^{N} \frac{D_n^2}{N} \approx E[D_n^2]$ is an unbiased and consistent estimator for $\bar{\sigma}^2$.

[23]Using one year of high frequency data, the 1994 year.

[24]The standard deviations are very high specially due to the sample size of 7 option prices.

[25]We also tested an estimator with lighter weights, $W_i = \frac{1}{P_i}$, but the results obtained were very similar to the results obtained for the WRP estimator.

[26]but not so close as in the previous studies performed using the SVFPS model.

[27] for instance, using different weighting schemes as suggested in Section 1.3

[28]General results regarding different functions f are not necessary for our analysis, and are hard to be derived.

[29]Plus the negligible term $\mu\sqrt{\Delta t}$.

[30]The calculations of the moments of this estimator were not provided by the authors of the model SVFPS.

Parameters	Value	Std
α	0.4760	0.0432
ν	0.3220	0.0097
c	0.4784	0.0069

Table 1: Estimated Parameters Using the Variogram Method.

Strike Prices	Call Price	Volume
20	4.30	418
22	2.70	418
24	1.42	3526
26	0.63	4046
28	0.25	2458
30	0.10	566
32	0.04	35

Table 2: Telemar Options Data on November 13, 2002, at 13:30.

Estimator	V_2	V_3
IV	-0.0248	0.0015
	(0.0083)	(0.0007)
OLS	-0.0178	0.0025
	(0.0048)	(0.0009)
WRP	-0.0033	-0.0011
	(0.0260)	(0.0040)

Table 3: Estimated Parameters Fitting Implied Volatility Including OM Data, Using Different Estimators.

Estimator	$SRMSE$
IV	0.0244
OLS	0.0454
WRP	0.0697

Table 4: A Measure to Compare the Different Estimators Fit to the Implied Volatility Including OM Data.

Estimator	V_2	V_3
IV	-0.0139	0.0034
	(0.0096)	(0.0007)
OLS	-0.0137	0.0037
	(0.0078)	(0.0011)
WRP	-0.0143	0.0026
	(0.0011)	(0.0003)

Table 5: Estimated Parameters Fitting Implied Volatility not Including OM Data, Using Different Estimators.

Estimator	$SRMSE$
IV	0.0060
OLS	0.0126
WRP	0.0222

Table 6: A Measure to Compare the Different Estimators Fit to the Implied Volatility not Including OM Data.

ρ	Mean: Estim. of $\frac{1}{\alpha}$	Std: Estim. of $\frac{1}{\alpha}$	Mean: Estim. of ν	Std: Estim. of ν
0	1.93	0.91	0.3280	0.0447
-0.1	1.91	1.01	0.3312	0.0462
-0.3	1.99	0.93	0.3293	0.0450
-0.5	1.85	0.97	0.3371	0.0505
-0.7	1.98	0.84	0.3308	0.0435
-0.8	1.90	0.94	0.3352	0.0437
-0.9	1.91	0.86	0.3355	0.0466

Table 7: Sensitivity of the Variogram Estimator to the Correlation Coefficient ρ.
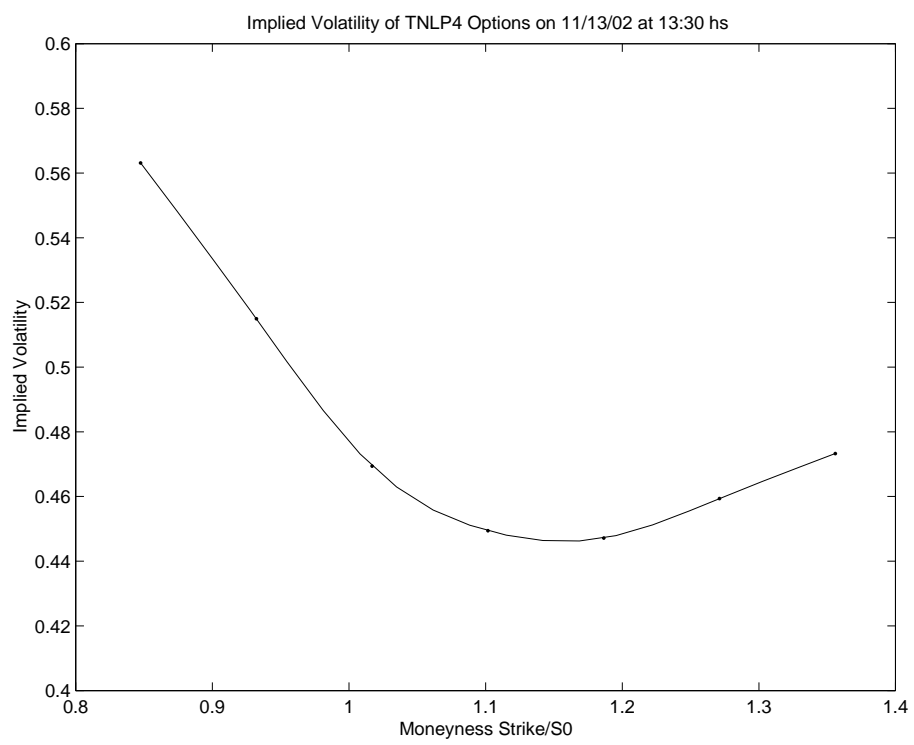
Figure 1: B&S Implied Volatilities as a Function of Moneyness.
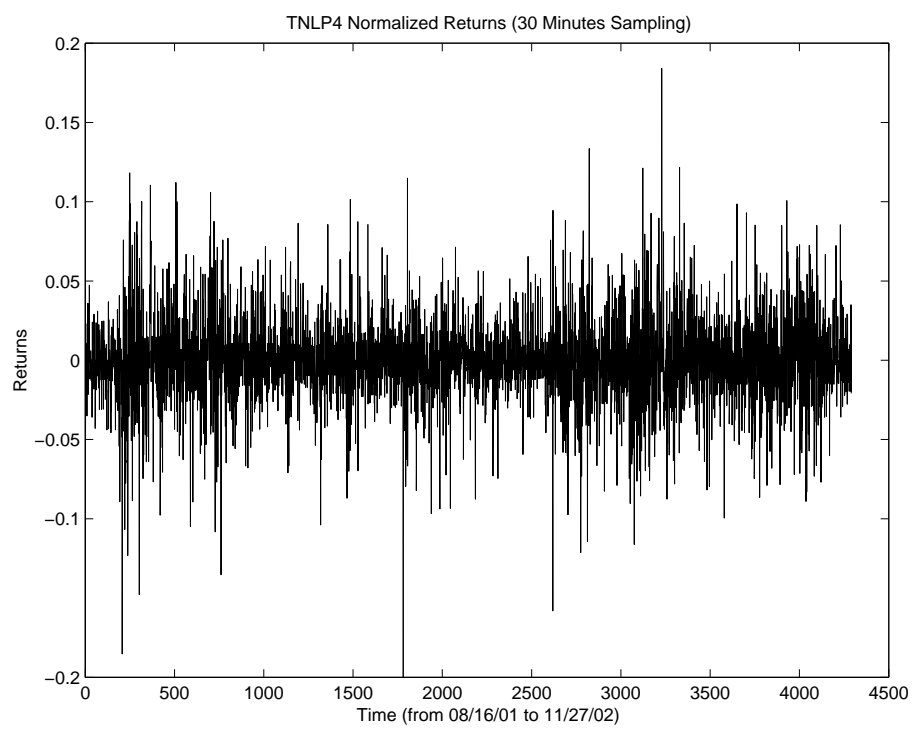
Figure 2: Amplitude of TNLP4 Fluctuations and Speed of Mean Reversion.
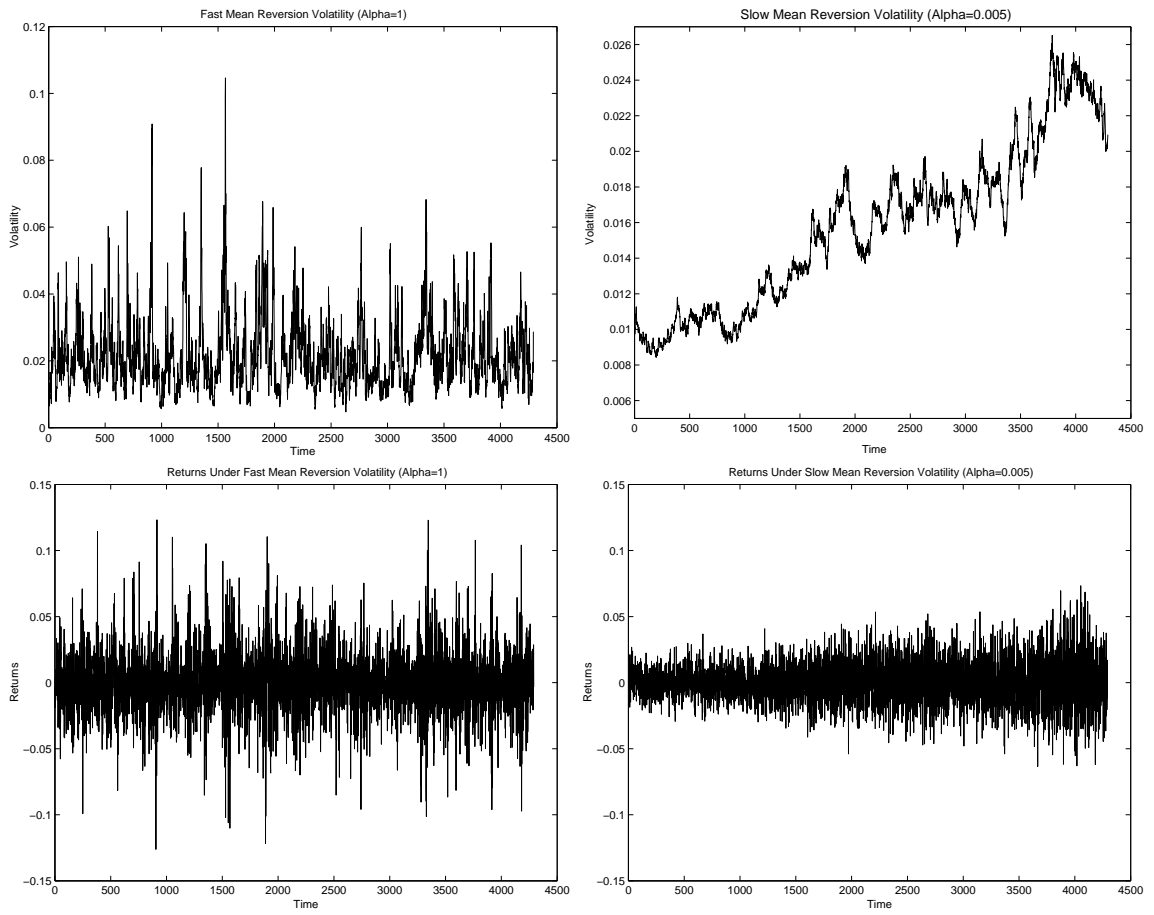
Figure 3: Simulated Volatility and Correspondent Returns for Small and Large Rates of Mean Reversion.
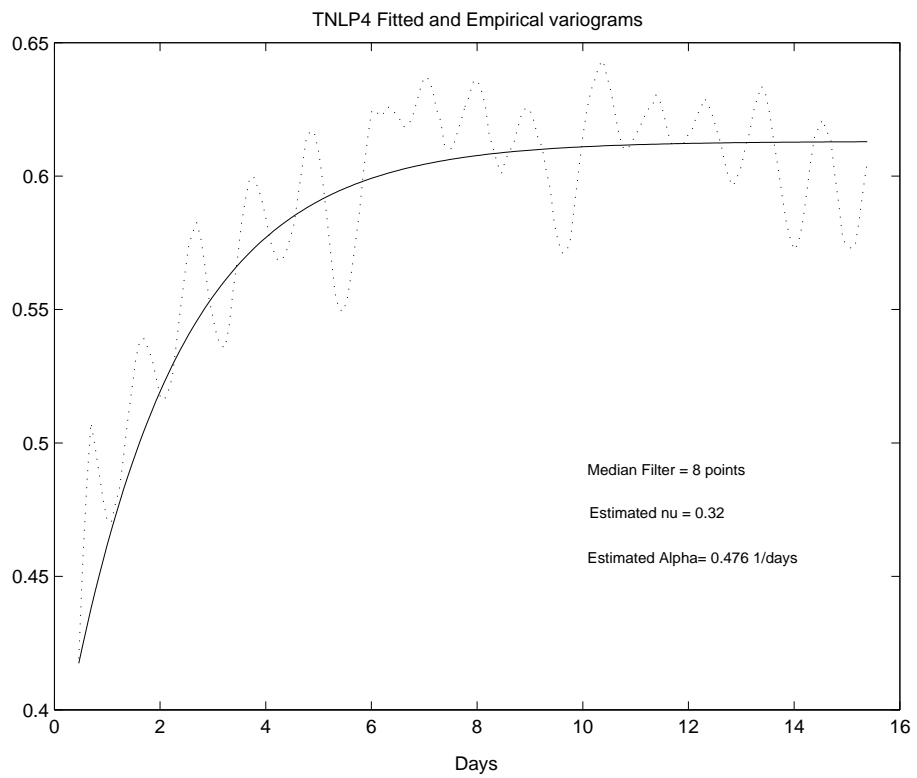
Figure 4: Estimating the Volatility Mean Reversion Speed.
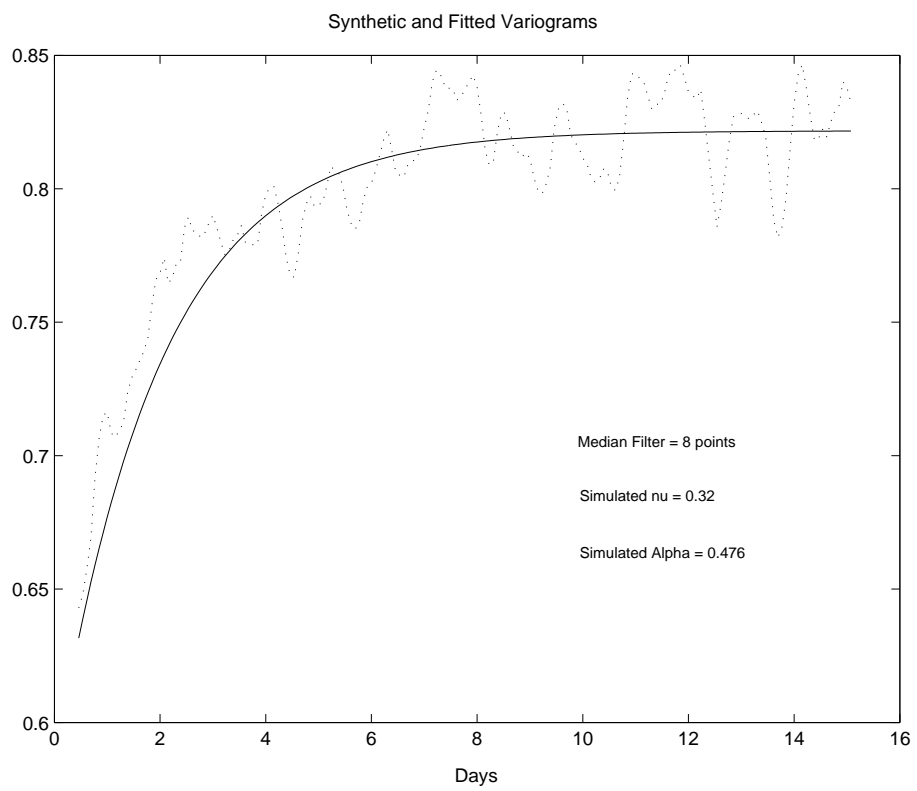
Figure 5: Mimicking the Empirical Variogram.
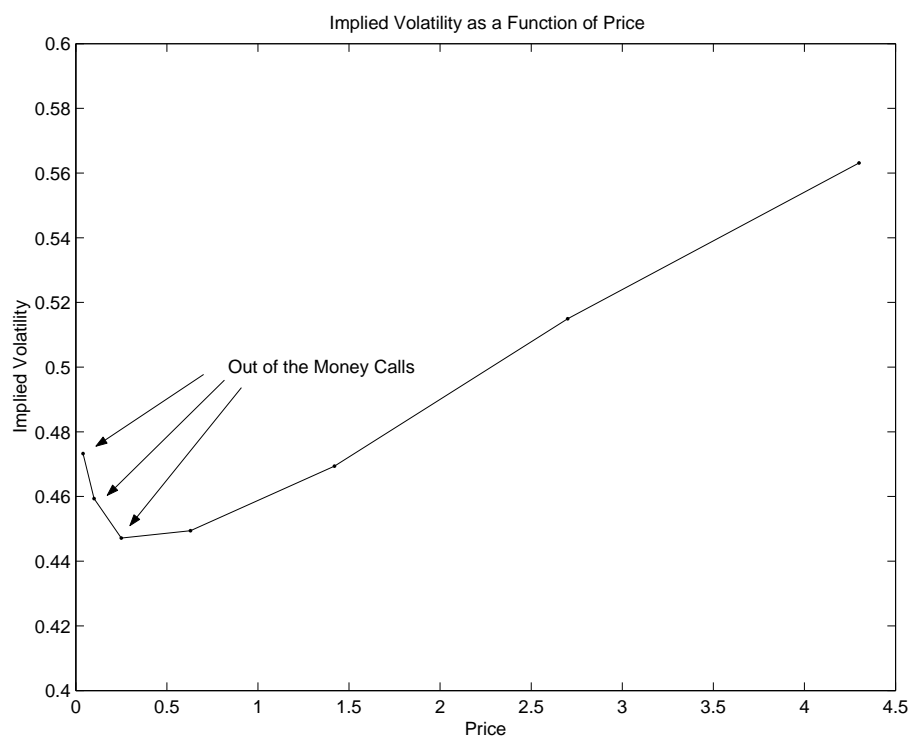
Figure 6: The Reason for Adopting a Better Approximation for the Model.
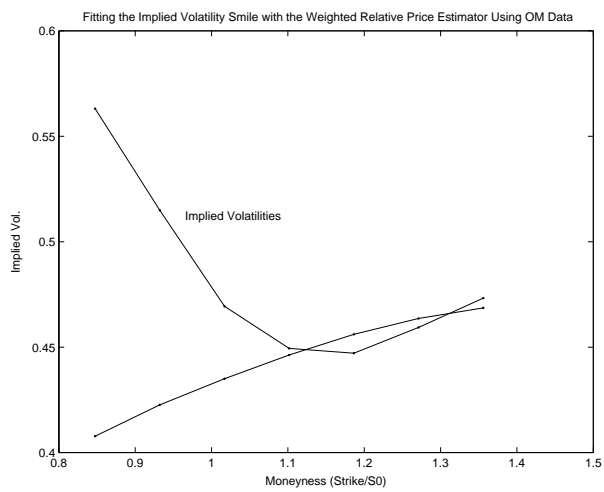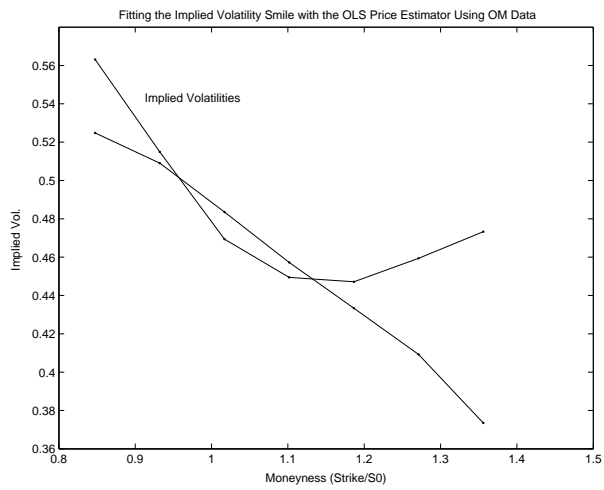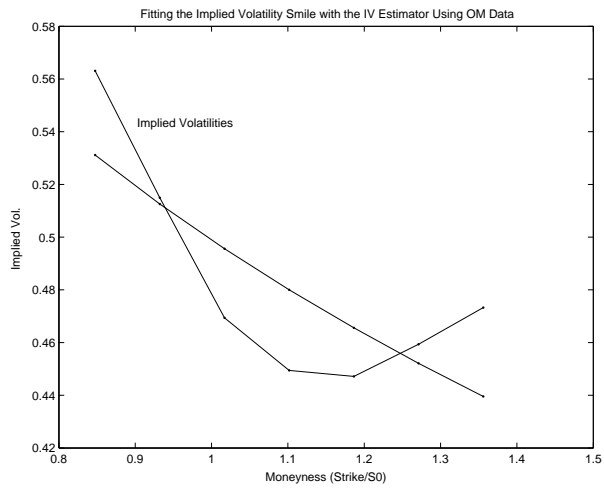
Figure 7: Implied Volatility Fit Including OM Data.
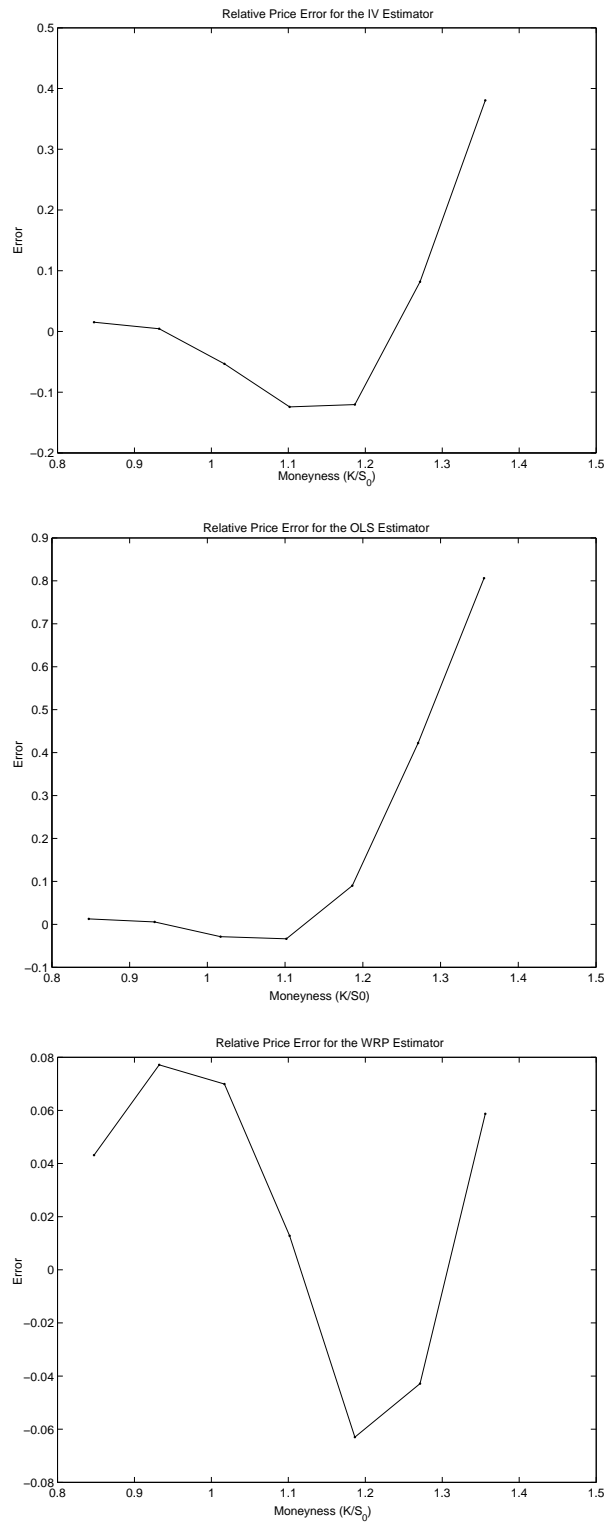
Figure 8: Relative Price Errors for the Three Estimators.
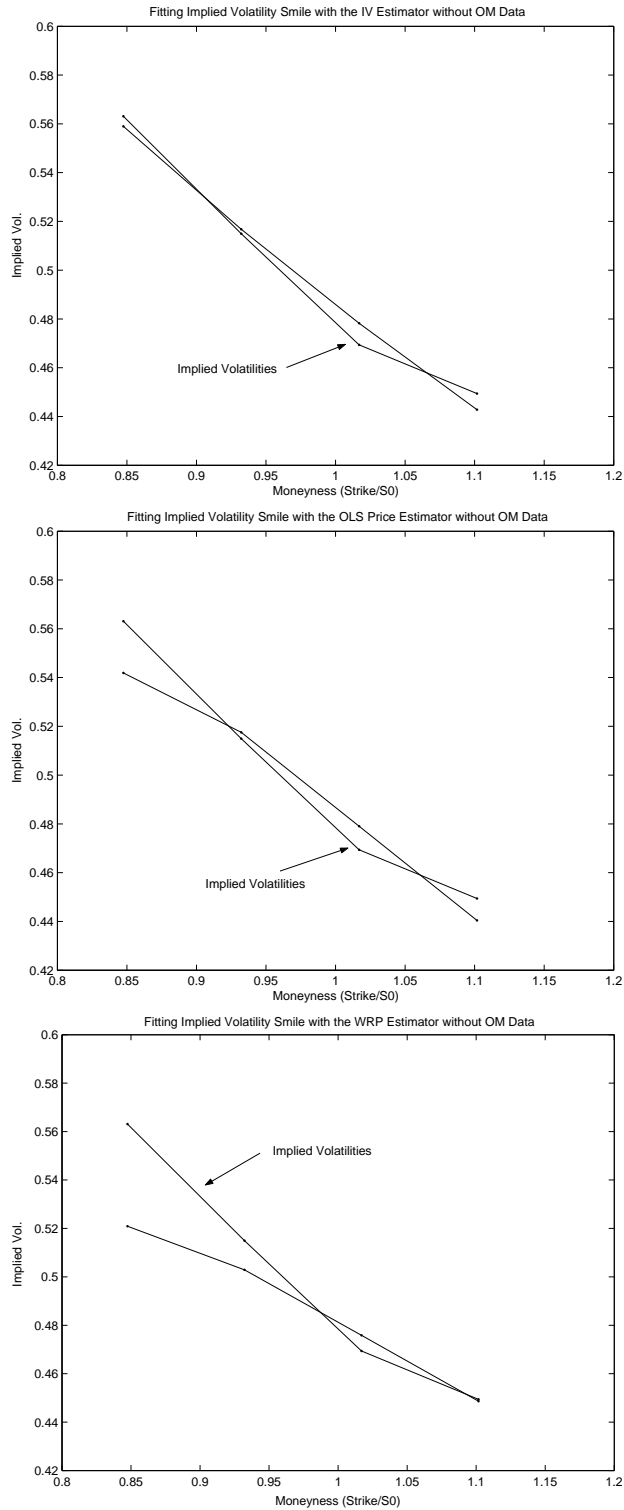
Figure 9: Implied Volatility Fit not Including OM Data.
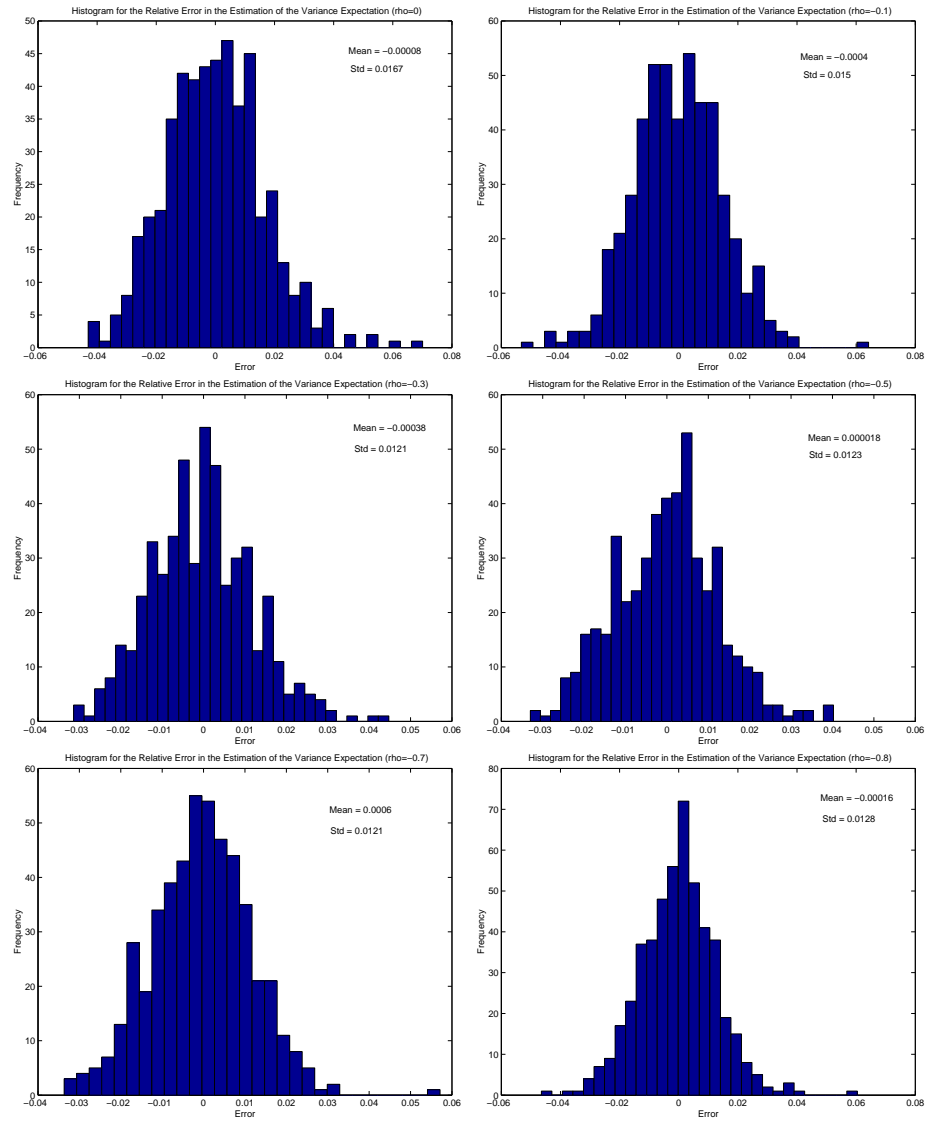
Figure 10: Histograms for the Relative Error in the Estimation of $\bar{\sigma}$ for different correlation coefficients.
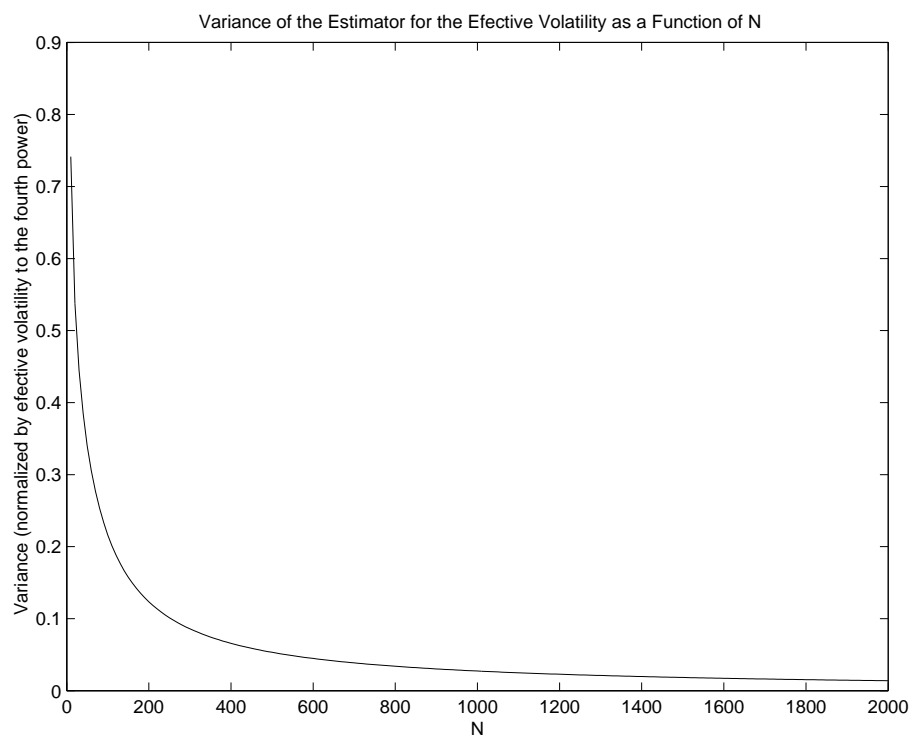
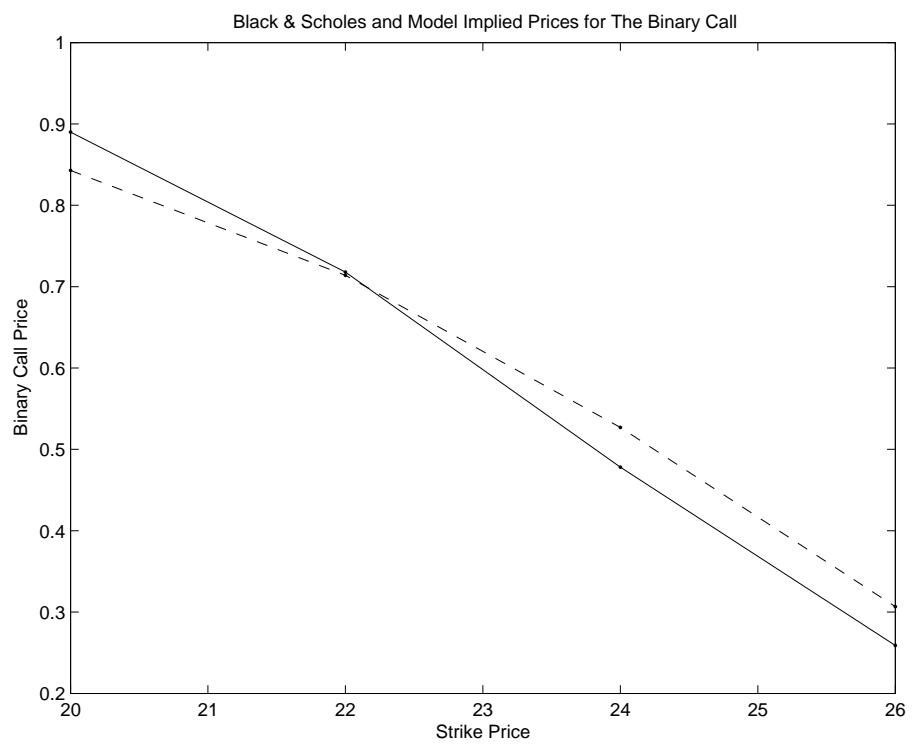Figure 11: Variance of the Estimator D_N as a function of the Number of Normalized Fluctuations.

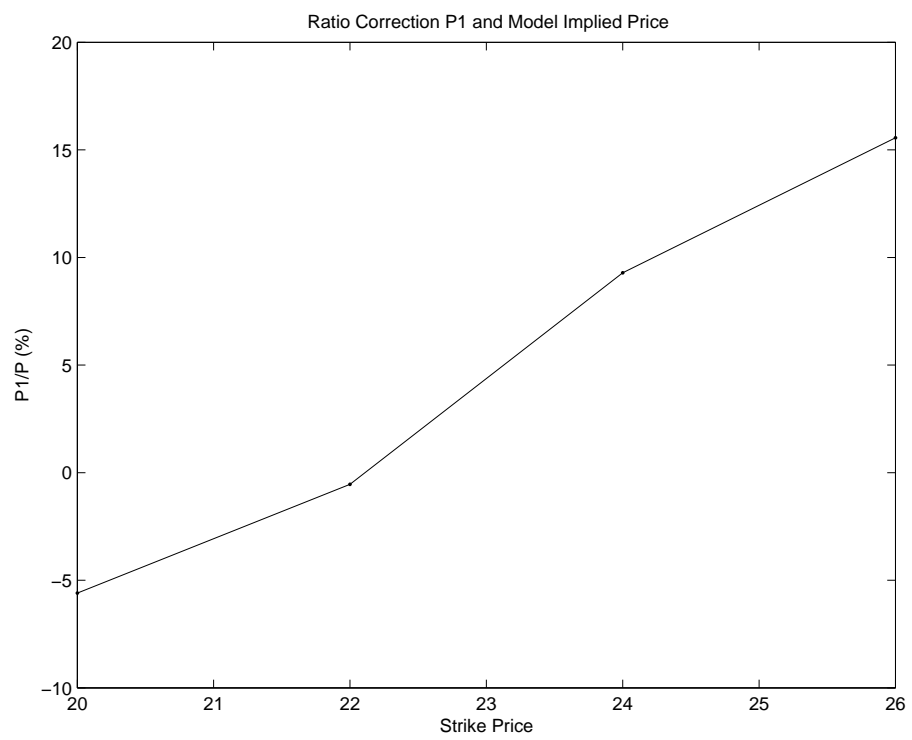Figure 12: Pricing a Binary Option Under Stochastic Volatility.

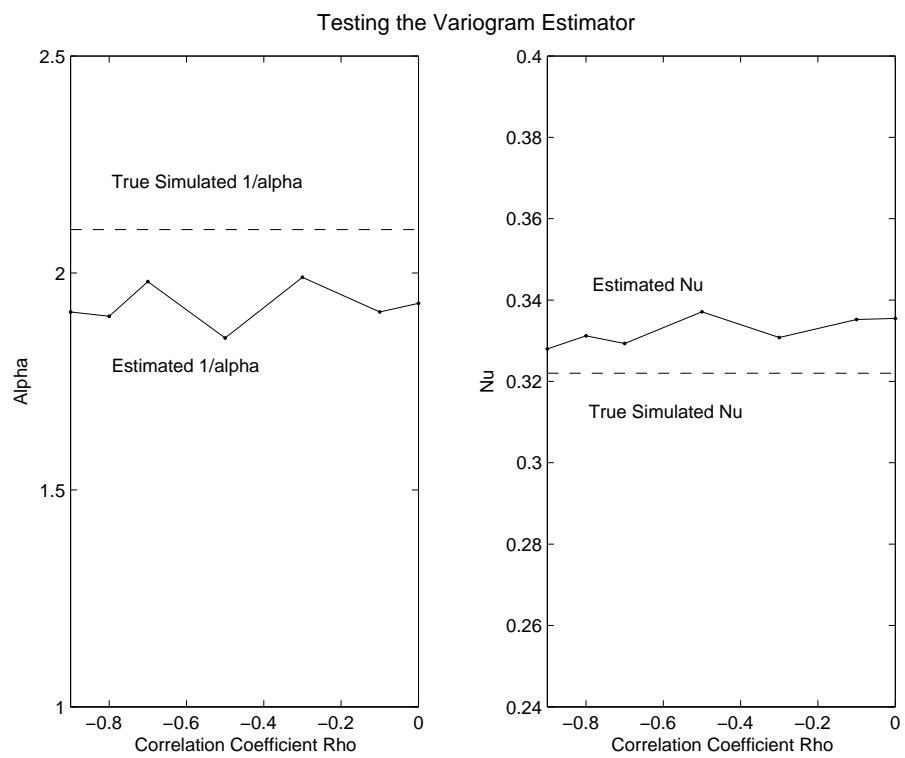Figure 13: The Amplitude of the Model Correction Term.

Figure 14: Sensitivity of the Variogram Estimators to Different Values of ρ.