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PRICING INTEREST RATE DERIVATIVES UNDER MONETARY CHANGES

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The goal of this paper is to develop a reduced-form model for pricing derivatives on the overnight rate. The model incorporates jumps around central bank (CB) meetings. More specifically, rate changes are decomposed into fluctuations between CB meetings and deterministic timed jumps following CB meetings. This approach is useful for practitioners, since it allows the extraction of expectations regarding central bank decisions embedded in liquid instruments, as well as the use of these expectations for the pricing of less liquid derivatives, such as options, in a consistent manner. We discuss applications to 30-Day Fed funds options and IDI options traded in Brazil.

Keywords: Overnight interest rate; deterministic timed jumps; interest rate derivatives.

1. Introduction

According to the Bank for International Settlements (BIS) semiannual OTC derivatives statistics report released on June 2017, interest rate contracts dominate OTC derivatives markets in notional terms, and consequently, dynamism in this segment drives overall activity. The notional amount of outstanding OTC interest rate derivatives rose from USD 368 trillion to USD 416 trillion in the first half of 2017, BIS (2017).

Among the myriad of products available, interest rate options and, in particular, interest rate caps play an important role in fixed-income markets. Thus, finding accurate ways to price these options is a recurrent topic in both industry and academia. Over the years, very sophisticated pricing models of interest rate options have appeared in the literature. A comprehensive review of the literature on interest rate modeling can be found in the book by Gibson *et al.* (2010), but despite the relatively complex mathematics involved, almost all models developed share the same foundation whereby the factor structure is invariably based on latent state variables.

On the one hand, latent factors modeling is undeniably an attractive method since it brings more flexibility to the model and improves its fitting accuracy. On the other hand, it is rather difficult to include an economic interpretation or an economic event to these latent factors in order to understand the forces driving option prices. Possibly, no single financial policy carries more weight than the Federal Reserve funds target rate, whose movements result from the decisions made by the Federal Open Market Committee (FOMC). Consequently, market participants use the price of a near-to-expiration derivative to recover the probability that a change in the target federal funds rate will be announced at the conclusion of the upcoming FOMC meeting.

The empirical literature about the predictability of monetary changes using derivatives is extensive. Starting with Ederington & Lee (1996) who analyze the response of options on USA Treasury, Eurodollar, and foreign exchange futures to a number of different macroeconomic announcements using an approach similar to Patell & Wolfson (1979, 1981). They find that the implied volatility increases on days without announcements and decreases after a wide range of macroeconomic announcements. Beber & Brandt (2006) find that the risk-neutral skewness and kurtosis embedded in treasury bond futures options change around scheduled macroeconomic announcements, in addition to documenting that the implied volatility decreases after the announcements.

Historically, practitioners have used LIBOR as a proxy for *risk-free* rates when valuing derivatives. However, recently the use of LIBOR as the *risk-free* rate to value derivatives was called into question by two major events. The first one, as pointed out by Hull & White (2013), has been the credit crisis in 2008, where banks became increasingly reluctant to lend to each other because of credit concerns. As a result, LIBOR quotes started to rise. The TED spread, which is the spread between three-month USA dollar LIBOR and the three-month USA Treasury rate, is less than 50 basis points (bps) in normal market conditions. According to the authors, between October 2007 and May 2009, it was rarely lower than 100 bps and peaked at over 450 bps in October 2008.

But LIBOR suffered a much more harmful hit on its credibility when the Wall Street Journal (WSJ) reported a marked divergence between the LIBOR rate and the WSJ's calculation of the rate in the default insurance market. Fouquau & Spieser (2015) show the chronology of facts and empirically support the claim that the LIBOR market was rigged. As a result, most derivatives dealers now use interest rates based on overnight indexed swap (OIS) rates rather than LIBOR when valuing collateralized derivatives. Conceptually, an OIS is an interest rate swap where the floating rate of the swap is equal to the geometric average of an overnight index rate over every day of the payment period. The index rate is typically a central bank rate or equivalent, for example the Federal funds rate in the USA.

By putting together these two elements, the overnight interest rate and meetings schedule of the monetary authority, we will subsequently attempt to answer the following question: can the inclusion of scheduled events improve the pricing of interest rate derivatives, and if so, to what extent? Thus, the goal of this paper is to develop a tractable reduced form model incorporating jumps on central bank meetings to price derivatives on overnight interest rate. The key element in our model, unlike traditional interest rate models, is the fact that we break down the overnight rate into two components, the first one is a continuous processes governing the overnight rate between two scheduled meetings and a second one formed by a deterministically timed jump describing central bank meetings outcome. Moreover, the fact that the monetary authority, in general, changes the target rate in 25-bps increments makes the inclusion of jumps much more straightforward than in the case of assets such as equities, foreign exchange, or commodities with discontinuous changes that are better modeled as realizations drawn from continuous distributions.

A simple extension of the Black (1976) model incorporating deterministic jumps provides the essence of our approach. This approach is particularly useful for practitioners because it allows them to extract expectations regarding central bank decisions that are embedded in liquid instruments and to use it to consistently price less liquid instruments such as interest rate options.

The rest of this paper is organized as follows. Section 2 presents the paper's motivation. Section 3 summarizes the literature on jumps. This review includes both standard random jumps and deterministic timed jumps. Section 4 presents a model to include deterministic timed jumps into interest rate modeling. Then, in Sec. 5, we describe how discrete time Markov chain (DTMC) can be used to include dependency among monetary decisions into our framework. Section 6 presents closed-form solutions for pricing zero-coupon bonds and European call and put options incorporating the market expectations about future changes in the monetary policy. Section 7 describes how practical applications of using interest rate futures for extracting implied market expectations about future changes in the monetary policy in the USA and Brazil. In Sec. 8, we assess the quality of the model proposed in this paper for pricing real market options traded in Brazil and in the USA. Finally, Sec. 9 presents our concluding remarks.

2. Motivation

Monetary policy implementation refers to the tools and practices that a central bank uses to achieve its policy objectives (Friedman 1968). By implementing effective monetary policy, the monetary authority can maintain stable prices, thereby supporting conditions for long-term economic growth and maximum employment. In a nutshell, the monetary authority has three instruments to attain its policy goals: open market operations, the discount rate and reserve requirements. The monetary authority sets interest rates either directly by changing the discount rate or through the use of open market operations by buying and selling government securities which affects the target rate.

In the USA, the FOMC is the branch of the Federal Reserve Board that determines the direction of monetary policy. The FOMC conducts eight scheduled meetings per year, approximately one each six weeks, and the schedule of meetings for a specific year is announced ahead of time. Most importantly, FOMC meetings are one of the key economic events for traders and investors alike. Although the central bank's policies are targeted towards controlling the short-term interest rates, they have far reaching implications across both the yield curve and stock market. Therefore, it goes without saying that the FOMC meetings are one of the volatile events on the economic calendar.

To motivate our model, we present the evolution of the overnight interest rate in the USA and in Brazil, as well as the target rate set by the monetary authority in each country. In Fig. 1, we present the evolution of the effective federal funds rate (EFFR) which is the interest rate at which depository institutions (banks and credit unions) lend reserve balances to other depository institutions overnight, it is calculated as a volume-weighted median of overnight federal funds transactions with domestic unsecured borrowings in USA dollars. Since January 2009, the FOMC sets a target range for the federal funds rate, which they enforce by open market operations and adjustments in the interest rate on reserves held at the Federal Reserve by depository institutions.

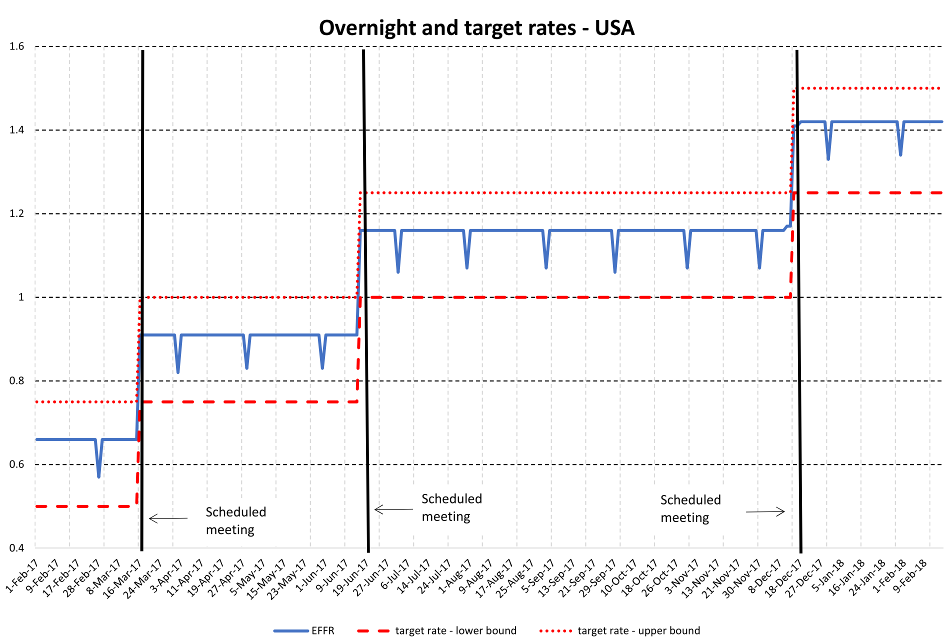

Fig. 1. The effective federal funds rate (EFFR) (blue line) and target rate/range (red line). Vertical lines are scheduled FOMC meetings.

As can be seen, the interbank borrowing rate lies within a range of that target rate set by the Federal Reserve and for the selected scheduled meetings of the FOMC we can observe the pronounced effect of jumps on overnight rate.

At the beginning of March 1999, in an environment still marked by uncertainty about the impact of the devaluation of the Brazilian currency (BRL) on inflation, the Brazilian government announced its intention to start conducting the monetary policy based on an inflation targeting (IT) framework.

Most central banks use a short-term interest rate as the main instrument of policy. Likewise, the central bank of Brazil (BCB) uses the Selic rate as the primary instrument of monetary policy. The Monetary Policy Committee (COPOM) conducts eight scheduled meetings per year where it sets the target for the Selic rate and delegates to the open market desk operations of the BCB to keep the effective Selic rate close to the target. Due to transmission channels in the financial market, the Selic target also steers the rate at which the Brazilian banks are willing to borrow/lend to each other in overnight unsecured transaction denominated in BRL, known as CDI rate. The equivalent of the CDI rate in the American market can be considered as the effective federal funds rate (EFFR). Figure 2 shows the actual overnight CDI rate and the Selic target in Brazil.

The Selic target is fixed for the period between its regular meetings and to avoid any potential criticisms about insider information the COPOM releases its decision

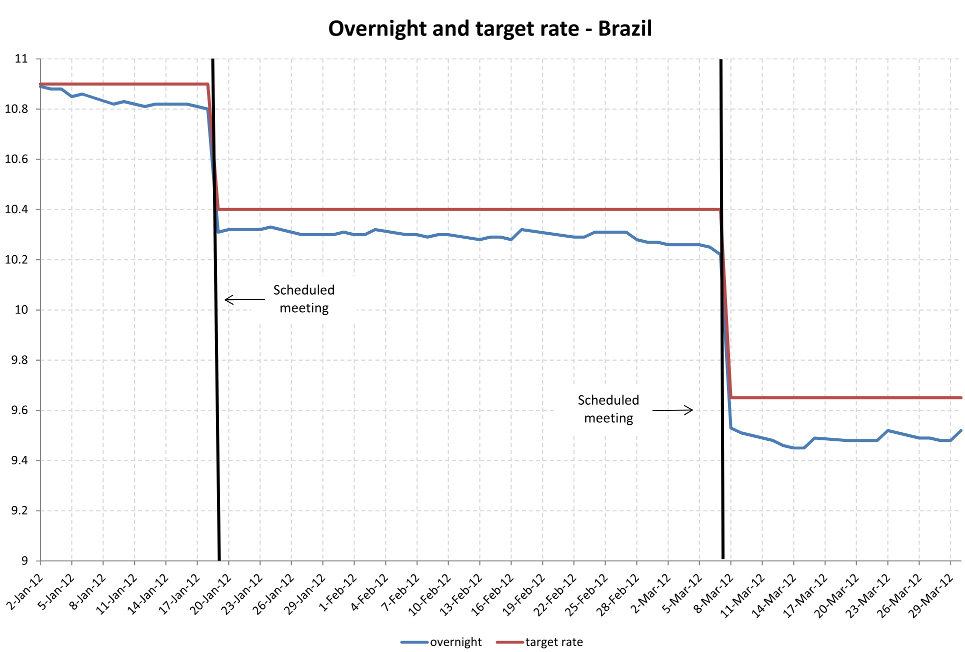

Fig. 2. Overnight interest rate (CDI) (blue line) and target rate (red line). Vertical lines are scheduled COPOM meetings.

when the Brazilian market is closed so that the overnight rate has a deterministically timed jump occurring at the day after the scheduled meeting.

These findings are in line with our assumption that the observable interest rate can be break down into two components: a continuous process describing the overnight rate between meetings and a point process which captures the monetary decisions.

3. Asset Prices with Jumps

There are many papers on the subject of incorporating jumps in the price process of stocks. Many of these articles model the jump occurrence as random. In this category of models, we have the seminal paper by Merton (1976). In this paper, the author extends the Black and Scholes model with a Poisson process to capture abnormal price variations that the normal Black and Scholes model does not. A more general jump model was proposed by Kou & Wang (2004) by assuming that jumps have a double exponential distribution instead of a Normal distribution as presented in Merton (1976). Both of these models assume that the occurrence and size for jumps are stochastic.

Even though the literature presents that the inclusion of jumps provides a better statistical characterization for equity prices due to considerable presence of skewness and kurtosis, as pointed out in Eraker (2004), the response of equity prices to earnings announcements is different from the unpredictable events in the models above because the timing of the release of the information is known in advance, although the response of the underlying price to the event is not. Thus, a natural extension is to assume that equity prices have a deterministically timed jump occurring at the earnings release. In Dubinsky & Johannes (2004), the authors propose a model where the timing of earnings announcements, although not the response of equity prices, is known in advance. To model the behavior of these events, i.e. earnings announcements, they develop two different jump models, one with constant diffusive volatility and deterministically timed jumps and one with stochastic volatility and deterministic jumps. The paper finds that accounting for jumps on earnings announcement dates is extremely important for pricing options. Models without jumps on earnings announcement dates have large and systematic pricing errors around earnings dates. A stochastic volatility model incorporating earnings jumps drastically lowers the pricing errors and reduces misspecification in the volatility process.

Earnings announcements are marked as special events for single-stock options traders due to their potential impact on the underlying asset, however the influence of scheduled events is not circumscribed to the stock market. In the USA, meeting days of the FOMC are carefully monitored by market participants, because FOMC announcements often cause strong reactions in bond and stock markets. These observations suggest that models for pricing interest rate instruments should take into account monetary policy actions by the Federal Reserve, and this channel is set forth in the work by Piazzesi (2005). The author develops an arbitrage-free time-series model of yields in continuous-time that incorporates central bank policy. In her model, the Federal Reserve's target rate is a pure jump process and jump intensities depend on the state of the economy and the meeting calendar of the FOMC. The author shows that her methodology improves the fit of the yield curve and introduces important seasonalities around FOMC meetings.

Although presenting advances in interest rate related literature, Piazzesi's (2005) model relies on a structural approach where some latent variables and parameters are unobservables which makes her model subject to different levels of model risk and thus of limited use for practitioners. Therefore, in this paper we contribute to the literature by proposing a tractable reduced-form model to price derivatives on overnight interest rate. The model breaks down short-term interest rate changes into fluctuations between meetings and deterministic timed jumps around central bank meetings.

4. Interest Rate Process with Scheduled Events

The uncertainty in our model is introduced through two stochastic processes: the continuous overnight interest rate $(r_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$ which reflect the changes in the target rate defined by the central bank in its scheduled meetings. Both processes are defined on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\mathcal{F}_t$ is the natural filtration generated by (r_t, θ_t).

To properly model deterministically timed events, we also assume that there is a deterministic counting process N_t, counting the number of predictable events that occur up to time t and define τ_j for the time of the jth meeting after a starting time t:

$$N_t = \sum_j \mathbb{I}_{\{\tau_j \leq t\}}, \tag{4.1}$$

where $(\tau_j)_{j\geq 1}$ are increasing predictable stopping times as defined in Brémaud (1981).

According to our model's assumptions, the observable overnight interest rate for a given time t is the result of two components, the first one is a continuous overnight rate $(r_t)_{t\geq 0}$ process which describes the overnight rate evolution between two central bank meetings and the second one captures monetary decisions, $(\theta_t)_{t\geq 0}$:

$$R_t = r_t + M_t. \tag{4.2}$$

where $M_t = \sum_{j=1}^{N_t} \theta_j$ is the sequence of changes in the target rate implemented by the monetary authority up to time t.

Therefore, we assume that the observable overnight interest rate is given by the stochastic differential equation (SDE) of the form

$$\mathrm{d}R_t = \mu(r_t)\mathrm{d}t + \sigma(r_t)\mathrm{d}W_t + \mathrm{d}M_t, \tag{4.3}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the usual conditions for existence and uniqueness for the solution of R_t (Theorem 5.2 in Øksendal (2000)), $(W_t)_{t\geq 0}$ is a one-dimensional

Brownian motion, $(M_t)_{t\geq 0}$ is a pure jump process with jumps at the scheduled events distributed as $\theta_j \sim \Psi$ and $(N_t)_{t\geq 0}$ is the deterministic counting process. The process $(R_t)_{t\geq 0}$ is right-continuous, adapted and has only a finite number of discontinuities in each interval $[0,t]$, processes M_t and W_t are also independents.

In our framework, Eq. (4.2) represents the observable overnight interest rate, R_t which is the sum of two terms: the overnight interest rate, r_t prevailing between two consecutive meeting τ_j and τ_{j+1} and the second term, $M_t = \sum_{j=1}^{N_t} \theta_j$ which describes the sequence of changes, θ_j in the target rate implemented by the monetary authority up to time t. The exact expression to characterize the dynamics of the observable overnight interest rate in (4.3) still requires the functional form for describing the overnight interest rate, r_t, between two consecutive meeting and this will be provided in Sec. 6.

Most applications of models with jumps assume that Ψ is a continuous distribution, such as the normal distribution in Merton (1976) and in Dubinsky & Johannes (2004) or the double exponential in Kou & Wang (2004). The main reason is that these papers are modeling the impact of jumps on stock prices and it is not possible to define *a priori* the magnitude of the jump, however when dealing with monetary decisions, the outcome can be defined in a discrete set $\mathcal{A}$. In practice we observe that θ_j assume values that are multiples of some known quantity, for instance 25 bps.

According to our model's assumptions, between two central bank meetings, the interest rate diffuses, that is, they have continuous sample paths with Brownian shocks. At a scheduled meeting (jump time, τ_j), the interest rate jumps by a random size θ_j. Our model does not include randomly timed jumps in prices for two reasons. First, we are primarily interested in the impact of scheduled meetings on option prices and, as history shows, there is a small probability of an emergency move outside of a scheduled meeting.[a] Second, if randomly timed jumps were included it would be necessary to set additional assumptions and consequently the model would be much more complex and less intuitive.

The main thrust of the paper is to provide a consistent way for pricing different interest rate derivative contracts taking into account policy-related events, such as FOMC meetings. Therefore, to price interest rate derivatives, we need a measure, $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the discounted price processes are martingales. The pricing approach is based on Piazzesi (2010). The martingale restriction requires the usual assumption that the drift of a tradable asset, $P(t,T)$, under $\mathbb{Q}$ has the risk-free rate of interest as the expected growth rate. This assumption ensures that between deterministic jump times, the discounted price process is a $\mathbb{Q}$-martingale. At a jump time, for prices to be a martingale, we require that $\mathbb{E}^{\mathbb{Q}}[P(\tau_j,T)\,|\,\mathcal{F}_{\tau_j-}] = P(\tau_j-,T)$ which implies that at a deterministic time, there can be no profit.

[a]In the USA, there have been seven changes outside of FOMC meetings from 1994 to 2017. Likewise, in Brazil since 1999, the central bank has modified the target rate in an extraordinary meeting only once in a total of 117 regular meetings.

Therefore, we can state that there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ with Radon–Nikodým derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^T \gamma_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \gamma_s \mathrm{d}s\right)\prod_{j=1}^{N_t} X_{\tau_j}. \tag{4.4}$$

To ensure the existence of the Radon-Nikodým derivative γ and X must satisfy mild regularity conditions. For the diffuse component we assume that γ_s satisfies the Novikov condition. For the jump part, we require that $X_{\tau_j} > 0$ and that $\mathbb{E}^{\mathbb{P}}[X_{\tau_j} \mid \mathcal{F}_{\tau_j-}] = 1$. These conditions can be simultaneously met if we assume the jump size in the density process is equal to the ratio of jump size densities

$$X_{\tau_j} = \frac{\Psi^{\mathbb{Q}}(\theta_{\tau_j}, \tau_j-)}{\Psi^{\mathbb{P}}(\theta_{\tau_j}, \tau_j-)}. \tag{4.5}$$

Unlike diffusion models where only the drift can change subject to regularity conditions, in a jump model there are virtually no constraints other than common support, since $\Psi^{\mathbb{Q}}$ and $\Psi^{\mathbb{P}}$ are both positive. Therefore, given the mild assumptions required, the change of measure for jump sizes occurring at deterministic times is extremely flexible, for instance we could assume that Ψ is a Normal distribution, such as in Dubinsky & Johannes (2004), but for our purpose we define that θ can only assume values defined in a discrete set $\mathcal{A}$.

In general, the presence of jumps generates an incomplete market, due to the inability to hedge the continuously distributed jumps. In a way, to perfectly hedge jumps, one requires as many hedging instruments as the cardinality of the jump size distribution. With normally distributed jumps, this requires an infinite number of hedging instruments. On the other hand in our framework the cardinality is by construction finite, $(\mathrm{card}(\mathcal{A}) < \infty)$. This feature circumscribes our analysis to the standard complete market framework where there is a unique martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ used to calculate all instruments described in this paper.

5. Modeling $(\theta_t)_{t\geq 0}$ as a DTMC

Despite their highly unpredictable durations and amplitudes, business cycles tend to follow a fairly repetitive pattern as the economy gradually progresses through stages rather than switching suddenly from boom to bust and back to boom. Business cycles are highly intertwined with the push and pull of the monetary cycle whereby central banks transition from stimulating the economy during recessions to tightening money supply when inflationary pressures build up. As noted earlier, the main mechanism adopted by central banks for implementing monetary policy is the management of short-term interest rates using a set of administered rates (i.e. target rate).

In Fig. 3, we show the evolution of the target rate in Brazil and the monetary decisions set forth by the Brazilian central bank from January 2010 up to January 2018. From the data it is possible to identify periods where the central bank of Brazil

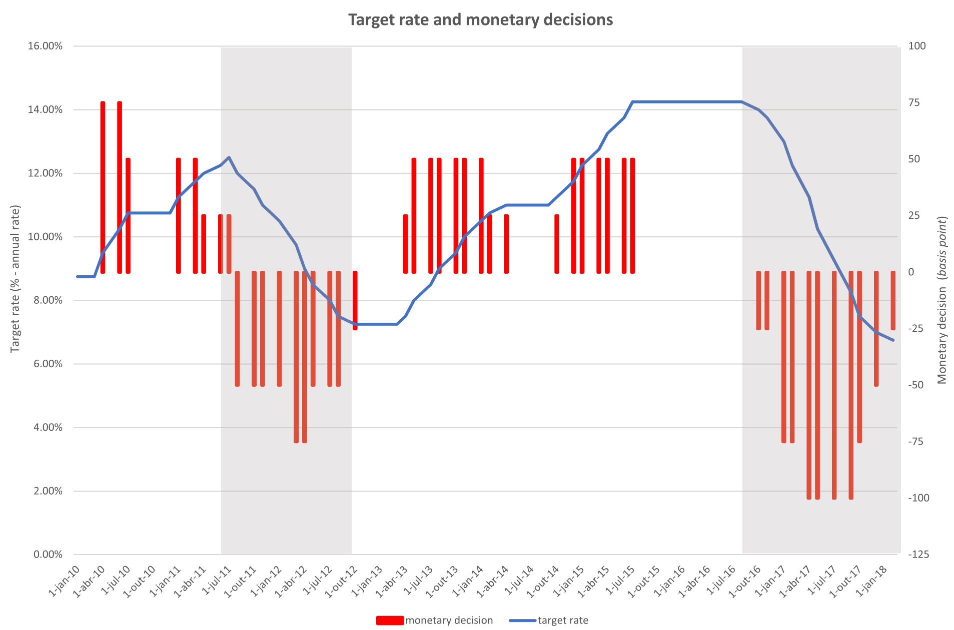

Fig. 3. Persistence on monetary decision in Brazil — sample from 2000Q1 to 2018Q1. Selic target rate (blue line — left axis) and monetary decisions on scheduled meeting (red bars — right axis). Shaded areas represent the Brazilian interest rate lossening-cycle.

increased the target rate and periods of monetary loosening, where the central bank decided to decrease the interest rate (the shaded areas).

Over the past 17 years the central bank's decision to hike the rate was the outcome in 24 scheduled meetings, τ_j (i.e. $\mathbb{P}(\theta_{\tau_j} > 0) = 0.37$), and the highest increase was 75 bps in the meeting held in April 2010. On the other hand, the central bank decreased the interest rate in 21 meetings ($\mathbb{P}(\theta_{\tau_j} < 0) = 0.32$), and the largest reduction was 100 bps. In turn, the interest rate was held unchanged in 20 scheduled meetings ($\mathbb{P}(\theta_{\tau_j} = 0) = 0.31$). Though the unconditional probabilities are uniform and therefore uninformative, the conditional probabilities provide a much more valuable source of information, $\mathbb{P}(\theta_{\tau_j} > 0 \,|\, \theta_{\tau_{j-1}} > 0) = 0.83$, $\mathbb{P}(\theta_{\tau_j} < 0 \,|\, \theta_{\tau_{j-1}} < 0) = 0.95$ and $\mathbb{P}(\theta_{\tau_j} = 0 \,|\, \theta_{\tau_{j-1}} = 0) = 0.75$. In other words, in a loose (tight) monetary cycle the probability of observing two reductions (increases) in a row is higher that two consecutive decisions with opposite signs.

Therefore, we do have elements to state that the process $(\theta_t)_{t\geq 0}$ itself depends on the monetary cycle pursued by the Central Bank and therefore is not temporally independent. A feasible way to incorporate simultaneously uncertainty and dependence on central bank decisions is by employing a discrete time Markov chain (DTMC) of order k for modeling $(\theta_t)_{t\geq 0}$.

A Markov chain with countably many states and transition matrix $\mathbf{P}$ with elements p_{si}, has transition probabilities $\mathbb{P}(\theta_{\tau_j} = a_i \,|\, \theta_{\tau_{j-1}} = s, \theta_{\tau_{j-2}} = s_{j-2}, \ldots, \theta_{\tau_0} =$

$s_0) = p_{si}$. The n-step transition matrix with elements $p_{si}^{(n)} = \mathbb{P}(\theta_{\tau_j+n} = a_i \,|\, \theta_{\tau_j} = s)$ equals $\mathbf{P}^n$.

For the sake of simplicity we assume that $(\theta_t)_{t\geq 0}$ is an ergodic Markov chain of order one. A Markov chain is called ergodic if there exists t such that for all $x, y \in \Omega$, $p_{xy}^{(t)} > 0$. For finite Markov chains, the following pair of conditions are equivalent to ergodicity:

(1) Irreducible: For all $x, y \in \Omega$, there exists $t = t(x, y)$ such that $p_{xy}^{(t)} > 0$.
(2) Aperiodic[b]: For all $x \in \Omega$, $\gcd\{t : p_{xx}^{(t)} > 0\} = 1$.

These assumptions are not too restrictive because: (i) one can always write a k-order DTMC as a first order DTMC, (ii) periodicity is not a rational behavior for a policy-maker following a Taylor-like policy rule, Taylor (1993) and (iii) the set $\mathcal{A}$ given by all potential values of central bank's decision about $(\theta_t)_{t\geq 0}$ is finite.

Usually θ_{τ_j} assume values that are multiples of some known quantity, for instance 25 bps. Therefore, we define $\mathcal{A}$ as the finite set of possible outcomes in one Central Bank meeting. Typical elements of $\mathcal{A}$ are $a = k \times 0.0025$ such that $k \in \mathbb{Z}$. Additionally, once θ_{τ_j} is DTMC its marginal distribution $\mathbb{P}(\theta_{\tau_j} = a_i)$ over $\mathcal{A}$ at time τ_j is described by[c]

$$\mathbb{P}(\theta_{\tau_{j+n}} = a_i) = \sum_s \mathbb{P}(\theta_{\tau_{j+n}} = a_i \,|\, \theta_{\tau_{j-1}} = a_s)\mathbb{P}(\theta_{\tau_{j-1}} = a_s) = \sum_s p_s p_{si}^{(n)}, \tag{5.1}$$

where transition probabilities $\mathbb{P}(\theta_{\tau_{j+n}} = a_i \,|\, \theta_{\tau_{j-1}} = a_s)$ satisfy the Chapman–Kolmogorov equation for n consecutive central bank meetings.

A convenient simplification arises in Eq. (5.1) when there is only one scheduled meeting ($n = 1$) before the derivative or bond maturity. In this case, $\theta_{\tau_{j-1}} \in \mathcal{F}_t$ given that $\tau_{j-1} \leq t$ and thus Eq. (5.1) simplifies to:

$$\mathbb{P}(\theta_{\tau_j} = a_i) = \mathbb{P}(\theta_{\tau_j} = a_i \,|\, \theta_{\tau_{j-1}} = a_s). \tag{5.2}$$

Such simplification is important to calibrate the transition probabilities from market prices.

6. Zero-Coupon Bond Pricing

The standard no-arbitrage theory states that a zero-coupon bond (ZCB) paying 1 at maturity T has price, $P(t, T)$, at time t given by:

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T R_s ds} \,|\, \mathcal{F}_t), \tag{6.1}$$

with boundary condition $P(T, T) = 1$.

[b]Here, gcd stands for the greatest common divisor.
[c]A technical question could arise when dealing with DTMC evolution. Equation (5.1) describes the probability for the process be at state $\theta_{\tau_j} = a, a \in \mathcal{A}$ after n steps. For our purpose, we might need the probability that the process hit by the first time the state $\theta_{\tau_j} = a \in \mathcal{A}$. Though conceptually different this distinction is not relevant when dealing with DTMC that walks few steps as in our case.

Duffie *et al.* (2003) shows that assuming the short-term interest rate process, R_s is affine implies that the zero-coupon bond prices must have the following form:

$$P(t,T) = e^{A(t,T)+B(t,T)R_t}, \tag{6.2}$$

with terminal conditions $A(T,T) = 0$ and $B(T,T) = 0$. The coefficients $A(t,T)$ and $B(t,T)$ can be computed in closed form for a few short-term interest rate processes, $(R_t)_{t\geq 0}$. When a closed form solution is not available it is still possible to obtain a solution numerically, for example by the Runge–Kutta method.

According to our model's assumptions, the observable short-term interest rate process, $(R_t)_{t\geq 0}$, is the result of two components, the first one is a continuous overnight rate $(r_t)_{t\geq 0}$ process which describes the overnight rate evolution between two central bank meetings and the second one captures monetary decisions, $(\theta_t)_{t\geq 0}$, thus:

$$R_t = r_t + M_t. \tag{6.3}$$

Here, $M_t = \sum_{j=1}^{N_t} \theta_j$ is the sequence of changes in the target rate implemented by the monetary authority up to time t.

Therefore, plugging Eq. (6.3) into Eq. (6.1), we have:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T R_s \mathrm{d}s} \,|\, \mathcal{F}_t) \tag{6.4}$$

$$= \mathbb{E}^{\mathbb{Q}}(e^{-(\int_t^T r_s \mathrm{d}s + \int_t^T M_s \mathrm{d}s)} \,|\, \mathcal{F}_t). \tag{6.5}$$

The actual evolution for the short-term rate in Brazil is shown in Fig. 2, where it is possible to identify the jumps when a COPOM meeting occurs and that between meetings the rate seems to fluctuate around a certain value without any large deviation regardless of the interest rate level. This type of behavior is present in models which include mean reversion and absence of level effect, i.e. the volatility of the interest rate increases with the level of the interest rate, Chan *et al.* (1992).

One specification for the short-term interest rate that satisfies these two empirical facts is the standard mean reversion Gaussian interest rate model developed initially by Vasicek (1977):

$$\mathrm{d}r_t = \kappa(\Theta - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{6.6}$$

where $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion, $\sigma > 0$ is the diffusion coefficient, Θ is the long-term interest rate level and $\kappa > 0$ is the speed of reversion.

Assuming that the overnight rate between scheduled meetings can be described by Eq. (6.6) it is possible to obtain an arbitrage-free formula for a zero-coupon bond that takes into consideration the future outcomes of the monetary authority.

Proposition 6.1. *The no-arbitrage price of a zero-coupon bond is given by*

$$P(t,T) = \sum_m (e^{-M_T + A(t,T) + B(t,T) r_t}) \mathbb{Q}(M_T = m), \tag{6.7}$$

where $A(t,T)$ and $B(t,T)$ are standard Vasicek coefficients given by

$$B(t,T) = -\frac{1-e^{-\kappa(T-t)}}{\kappa}, \tag{6.8}$$

$$A(t,T) = \left(\Theta - \frac{\sigma^2}{2\kappa^2}\right)[B(t,T)-(T-t)] - \frac{\sigma^2 B(t,T)^2}{4\kappa}, \tag{6.9}$$

and $\mathbb{Q}(M_T = m)$ are risk-neutral probabilities calculated by Eq. (5.1) and N_T is the number of scheduled meetings up to the bond maturity. The component $M_T = \sum_{j=1}^{N_T}\theta_j$ captures all monetary policy decisions at scheduled meetings over the interval $[t,T]$.

Proof of Proposition 6.1. The proof is quite straightforward, it starts by rewriting Eq. (6.4) as

$$\begin{aligned} P(t,T) &= \mathbb{E}^{\mathbb{Q}}(e^{-(\int_t^T r_s \mathrm{d}s + \int_t^T M_s \mathrm{d}s)} \mid \mathcal{F}_t) \\ &= \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}(e^{-(\int_t^T r_s \mathrm{d}s + \int_t^T M_s \mathrm{d}s)} \mid \mathcal{F}_t, M_T = m)]. \end{aligned} \tag{6.10}$$

By definition M_T is a discrete random variable, because $M_T = \sum_{j=1}^{N_T}\theta_j$ and $\theta_j \in \mathcal{A}$ with $\mathrm{card}(\mathcal{A}) < \infty$. Therefore, the outer expectation becomes

$$P(t,T) = \sum_m e^{-M_T}\mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T r_s \mathrm{d}s} \mid \mathcal{F}_t, M_T = m)\mathbb{Q}(M_T = m), \tag{6.11}$$

where the remaining expectation can be calculated using the framework of Duffie *et al.* (2003) by the fact that the Vasicek model is affine and has a closed form solution for $A(t,T)$ and $B(t,T)$, so Eq. (6.11) becomes

$$P(t,T) = \sum_m (e^{-M_T + A(t,T) + B(t,T)r_t})\mathbb{Q}(M_T = m). \tag{6.12}$$

□

So conditioning on M_T we can solve (6.10) as a classical ZCB pricing in a Vasicek model with a deterministic time-dependent drift. Finally, the ZCB price is obtained by calculating over all possible values of M_T weighted by their probability. By introducing time-dependent parameters in the model, we can match the current market's assessment of the future target rate, while retaining the overall simplicity of the term-structure model. This approach is advocated by, especially, Hull & White (2013) who extend the Vasicek and CIR models (Cox *et al.* 1985) with time-dependent parameters.

6.1. *Options pricing*

Interest rate futures are used by investors, either to hedge against interest rate changes or to assume outright positions. In addition, futures contracts are in general very liquid, which allow investors to trade their expectations about the future decisions of the monetary authority. On the other hand, interest rate options are not so liquid but their variety of payoff functions makes this derivative suited for some trading strategies such as all-or-nothing payoff.

For instance, binary options provide market participants the right payoff to trade on central bank futures decisions about the target rate. Binary options pay one unit of cash if the overnight interest rate R_t is equal or above the strike at maturity. Binary options are generally considered "exotic" instruments and there is no liquid market for trading these instruments between their issuance and expiration. The lack of liquidity to unwind a position before the maturity makes binary options less appealing in practice, because sometimes traders may need to adjust their position after a new economic indicator, which may impact central bank decision on $(\theta_t)_{t\geq 0}$, is released.

As observed earlier, part of the USA interest market has switched to overnight interest rate derivatives, such as overnight indexed swaps, OIS. An overnight indexed swap (OIS) is an interest rate swap where the periodic floating rate of the swap is equal to the geometric average of an overnight index rate over every day of the payment period. In addition to swaps, other derivatives can have overnight rates as their underlying such as options, for instance. In fact, we can point out IDI options traded at the Brazilian securities and derivatives exchange, B3, as an example of overnight indexed option. For more details about IDI options, see Brace (2008) or Carreira & Brostowicz (2016).

The underlying asset for IDI options is the IDI index defined as the accumulated overnight interest rate $(R_t)_{t\geq 0}$. Therefore, if we associate the continuously-compounded overnight interest rate to $(R_t)_{t\geq 0}$, then the IDI index is given by

$$\mathrm{IDI}_T = \mathrm{IDI}_t e^{\int_t^T R_s \mathrm{d}s}. \tag{6.13}$$

As can be seen, IDI options have a peculiar feature which is not shared by usual exchanged-traded options: they are asian options, and their payoff depends on the integral of the short-term rate through the path between the trading date t and the option maturity date T which will be impacted by the decision from the monetary authority that will take place on a scheduled meeting at time τ_j, where $t < \tau_j < T$.

Theoretical results of interest rate Asian options can be found in Geman & Yor (1993) and Longstaff (1995). Pricing IDI options was recently studied by Almeida & Vicente (2012) by specifying the overnight rate process, $(r_t)_{t\geq 0}$ as a sum of N processes with $\Theta = 0$ for all N in (6.6). But none of these papers acknowledge the role played by scheduled meetings, thus our model complements the above mentioned works.

An intermediate result relevant for pricing overnight interest rate options is the next lemma:

Lemma 1. *If* $(r_t)_{t\geq 0}$ *is a Vasicek process then*:

(1) *Distribution of integrated process*

$$\int_t^T r_s \mathrm{d}s \sim \; Normal(M(t,T), V(t,T)).$$

(2) *Zero-coupon bond price*

$$P(t,T) = e^{-M(t,T)+\frac{V(t,T)}{2}}, \tag{6.14}$$

where

$$M(t,T) = \frac{r_t - \Theta}{\kappa}(1 - e^{-\kappa(T-t)}) - \kappa(T-t), \tag{6.15}$$

$$V(t,T) = \frac{\sigma^2}{2\kappa^3}(2\kappa(T-t) - 3 + 4e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}), \tag{6.16}$$

parameters above have the same definition as in Eq. (6.6), *i.e.* $\sigma > 0$ *is the diffusion coefficient,* Θ *is the long-term interest rate level and* $\kappa > 0$ *is the speed of reversion. We do not prove this lemma because its proof is well known.*[d]

Denote by $\mathrm{Call}(T, K, R_t)$ the time t price of a call option on the IDI, with maturity T and strike price K. Then

$$\mathrm{Call}(T,K,R_t) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T R_s \mathrm{d}s}(\mathrm{IDI}_T - K)^+ \mid \mathcal{F}_t]. \tag{6.17}$$

Therefore, assuming that the observable short-term interest rate process, $(R_t)_{t\geq 0}$, is, as Eq. (6.3) implies, the result of two components, the first one describing monetary decisions, $(\theta_t)_{t\geq 0}$ and the second one a continuous overnight rate $(r_t)_{t\geq 0}$ given by Eq. (6.6) which describes the overnight rate evolution between two central bank meeting, it is therefore possible to obtain an arbitrage-free formula for pricing IDI that takes into consideration the future outcomes of the monetary authority.

Proposition 6.2. *The no-arbitrage price for a European IDI call option is given by*

$$\begin{aligned}\mathrm{Call}(T,K,R_t) &= \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T R_s \mathrm{d}s}(\mathrm{IDI}_T - K)^+ \mid \mathcal{F}_t] \\ &= \sum_m BS^{\star}_{\mathrm{call}}((r_t \mid M_T = m), \hat{K}_i, T, V(t,T), P(t,T))\mathbb{Q}(M_T = m),\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}&BS^{\star}_{\mathrm{call}}((r_t \mid M_T = m), K, T, V(t,T), P(t,T)) \\ &\quad = \mathrm{IDI}_t N(d_1) - \hat{K}_m P(t,T) N(d_2),\end{aligned} \tag{6.19}$$

$$d_1 = \frac{\log \dfrac{\mathrm{IDI}_t}{\hat{K}_m} + \log P(t,T) + \dfrac{V(t,T)}{2}}{\sqrt{V(t,T)}}, \tag{6.20}$$

$$d_2 = d_1 - \sqrt{V(t,T)}, \tag{6.21}$$

[d]However, the interested reader can consult Mamon (2004).

with

$M_T = \sum_{j=1}^{N_T} \theta_j$ *capturing all monetary policy decisions at scheduled meetings over the interval* $[t, T]$,

$\hat{K} = Ke^{-M_T}$ *is the corrected strike price*,

$P(t,T)$ *and* $V(t,T)$ *as in* (6.14) *and* (6.16), *respectively.*

Proof of Proposition 6.2. The proof consists of applying the tower property of conditional expectation, but first expression (6.17) can be simplified after plugging (6.3) and (6.13):

$$\begin{aligned}\mathrm{Call}(T,K,R_t) &= \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T R_s \mathrm{d}s}(\mathrm{IDI}_T - K)^+ \mid \mathcal{F}_t)\\ &= \mathbb{E}^{\mathbb{Q}}(e^{-\int_t^T (r_s \mathrm{d}s + M_s \mathrm{d}s)}(\mathrm{IDI}_t e^{\int_t^T (r_s \mathrm{d}s + M_s \mathrm{d}s)} - K)^+ \mid \mathcal{F}_t)\\ &= \mathbb{E}^{\mathbb{Q}}((\mathrm{IDI}_t - e^{-\int_t^T r_s \mathrm{d}s} K e^{-\int_t^T M_s \mathrm{d}s})^+ \mid \mathcal{F}_t). \end{aligned} \tag{6.22}$$

As stated earlier, $M_s = \sum_{j=1}^{N_s} \theta_j$ captures all monetary policy decisions at scheduled meetings over the interval $[t, s]$, so using the tower property of conditional expectation we can write:

$$\mathrm{Call}(T,K,R_t) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}((\mathrm{IDI}_t - e^{-\int_t^T r_s \mathrm{d}s} K e^{-\int_t^T M_s \mathrm{d}s})^+ \mid \mathcal{F}_t, M_T = m)]. \tag{6.23}$$

By definition M_T is a discrete random variable, because $M_T = \sum_{j=1}^{N_T} \theta_j$ and $\theta_j \in \mathcal{A}$ with $\mathrm{card}(\mathcal{A}) < \infty$. Therefore, the outer expectation becomes

$$\begin{aligned}&\mathrm{Call}(T,K,R_t)\\ &\quad = \sum_m \mathbb{E}^{\mathbb{Q}}((\mathrm{IDI}_t - e^{-\int_t^T r_s \mathrm{d}s} K e^{-\int_t^T M_s \mathrm{d}s})^+ \mid \mathcal{F}_t, M_T = m)\mathbb{Q}(M_T = m).\end{aligned} \tag{6.24}$$

The expectation can be solved using Lemma 1 and the solution follows as in Almeida & Vicente (2012) but here with modified strike price $\hat{K}$ which depends explicitly on M_T and $\Theta \neq 0$:

$$\mathrm{Call}(T,K,R_t) = \sum_m (\mathrm{IDI}_t N(d_1) - \hat{K}_m P(t,T) N(d_2))\mathbb{Q}(M_T = m). \tag{6.25}$$

□

This strategy of conditioning on all possible values of M_T is conceptually equivalent to Merton (1976) to price options when random jumps are present. Curran (1994) also adopted a similar strategy for pricing options where the underlying asset is the weighted arithmetic average value of a stock's portfolio.

If $\mathrm{Put}(T,K,R_t)$ is the price at time t of the IDI put with strike K and maturity T, then by the put-call parity we state the next result without proof.

Proposition 6.3. *The no-arbitrage price for a European IDI put option is given by*

$$\begin{aligned}\mathrm{Put}(T,K,R_t) &= \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T R_s \mathrm{d}s}(\mathrm{IDI}_T - K)^+ \mid \mathcal{F}_t]\\ &= \sum_m BS^{\star}_{\mathrm{put}}((r_t \mid M_T = m), \hat{K}_i, T, V(t,T), P(t,T))\mathbb{Q}(M_T = m),\end{aligned} \tag{6.26}$$

where

$$BS^{\star}_{\mathrm{put}}((r_t \mid M_T = m), K, T, V(t,T), P(t,T)) = \hat{K}_m P(t,T)N(-d_2) - \mathrm{IDI}_t N(-d_1), \tag{6.27}$$

and $P(t,T)$, $d_{1,2}$ and $V(t,T)$ are defined as stated at Proposition 6.2.

The results presented in the above propositions were obtained assuming that the interest rate is described by a Vasicek process. The adoption of a different model for the overnight interest rate is straightforward, the difference would be in terms $BS^{\star}_{\mathrm{call}}$ and $BS^{\star}_{\mathrm{put}}$.

Let us now consider the general case of pricing an interest rate contingent claim $V(T,R_T)$, in a model with a stochastic short-term interest rate process $(R_t)_{t\geq 0}$ subject to monetary policy changes. From the general theory we know that the price at t of $V(T,R_T)$ is given by the formula

$$V(t,T,R_t) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T R_s \mathrm{d}s} V(T,R_T) \mid \mathcal{F}_t]. \tag{6.28}$$

The expectation in Eq. (6.28) is difficult to evaluate because in order to compute the expected value we have to obtain the joint distribution (under $\mathbb{Q}$) of the two stochastic variables (the integral of R_t and $V(R_T,T)$) and finally integrate with respect to that distribution. Additionally, we also need to include the monetary policy changes. However, we can evaluate the expectation in Eq. (6.28) by using $P(t,T)$ as the numeraire. In this case the equivalent martingale measure associated with using $P(t,T)$ as the numeraire is the T*-forward measure* pioneered independently by Geman (1989) and Jamshidian (1989). This trick allows us to present a more general version of Propositions 6.2 and 6.3.

Proposition 6.4. *An option with integrable claim payoff $V \in L^1(\mathbb{Q}, \mathcal{F}_T)$ is priced at time t as*

$$V(t,T,R_t) = \sum_m [P(t,T,m)\mathbb{E}^{\mathbb{Q}_T}[V(T,R_T,m) \mid \mathcal{F}_t, M_T = m]]\mathbb{Q}(M_T = m), \tag{6.29}$$

where the T-forward measure $\mathbb{Q}_T$ is equivalent to $\mathbb{Q}$.

Proof of Proposition 6.4. Using $P(t,T)$ as the numeraire, Geman (1989) and Jamshidian (1989) show that Eq. (6.28) can be written as:

$$V(t,T,R_t) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T R_s \mathrm{d}s} \mid \mathcal{F}_t]\mathbb{E}^{\mathbb{Q}_T}[V(T,R_T) \mid \mathcal{F}_t]. \tag{6.30}$$

The expectation under $\mathbb{Q}$ of a product of two terms in Eq. (6.28) is converted into the product of two expectations in Eq. (6.30), one under $\mathbb{Q}$ and the other under the *T-forward measure* $\mathbb{Q}_T$, which is easier to evaluate. Using Eq. (6.3) and resorting to the method of conditioning on all possible values of M_T, Eq. (6.30) became

$$V(t,T,R_t) = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[e^{-(m+\int_t^T r_s \mathrm{d}s)} \mid \mathcal{F}_t, M_T = m]] \times \mathbb{E}^{\mathbb{Q}_T}[\mathbb{E}^{\mathbb{Q}_T}[V(T, r_T, m) \mid \mathcal{F}_t, M_T = m]], \tag{6.31}$$

where $M_T = \sum_{j=1}^{N_T} \theta_j$, in addition M_T is a discrete random variable, because $\theta_j \in \mathcal{A}$ with card $(\mathcal{A}) < \infty$. Therefore, the outer expectation becomes

$$V(t,T,R_t) = \sum_m \mathbb{E}^{\mathbb{Q}}[e^{-(m+\int_t^T r_s \mathrm{d}s)} \mid \mathcal{F}_t, M_T = m]\mathbb{Q}(M_T = m) \times \sum_m \mathbb{E}^{\mathbb{Q}_T}[V(T, r_T, m) \mid \mathcal{F}_t, M_T = m]\mathbb{Q}_T(M_T = m). \tag{6.32}$$

To complete the proof, first we need to use the Radon–Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}}$ to change from $\mathbb{Q}$ to $\mathbb{Q}_T$, which can be written as

$$\mathbb{Q}_T(A) = \int_A \frac{e^{-\int_0^T r_u \mathring{d}u}}{P(t,T)} \mathrm{d}\mathbb{Q}. \tag{6.33}$$

Then, defining $A = \{M_T = m\}$ and using the fact for a random variable Y it is possible to write $\mathbb{E}^{\mathbb{Q}}[Y \mathbb{1}_{\{M_T=m\}}] = \mathbb{E}^{\mathbb{Q}}[Y \mid M_T = m]\mathbb{Q}(M_T = m)$. Recognizing the variable Y as the Radon–Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{Q}}$ we can solve Eq. (6.33) and so rewrite Eq. (6.32) only in term of $\mathbb{Q}$. The final step uses the no-arbitrage pricing formula for a zero-coupon bond, $P(t,T,m) = \mathbb{E}^{\mathbb{Q}}[e^{-(m+\int_t^T r_s \mathrm{d}s)} \mid \mathcal{F}_t, M_T = m]$ and after grouping terms the result holds. □

If we use $V(T, R_T) = (IDI_T - K)^+$, where IDI_T is the forward value for the current IDI_t, we have that under the *T-forward measure*, the forward IDI index is a martingale, so applying the results from Proposition 6.4 we can obtain the Black (1976) formula adjusted for the presence of scheduled meetings of the monetary authority as presented in the next proposition.

Proposition 6.5 (Black with deterministic timed jump). *The modified Black* (1976) *model for pricing a IDI call option under monetary changes is given by*

$$\mathrm{Call}(T,K,R_t) = \sum_m [P(t,T,m)(\mathrm{IDI}_T N(d_1) - KN(d_2))]\mathbb{Q}(M_T = m), \tag{6.34}$$

where

$$\mathrm{IDI}_T = \frac{\mathrm{IDI}_t}{P(t,T,m)}, \tag{6.35}$$

$$d_1 = \frac{\log \frac{\mathrm{IDI}_T}{K} + 0.5\sigma_{\mathrm{IDI}}^2(T-t)}{\sigma_{\mathrm{IDI}}\sqrt{T-t}}, \tag{6.36}$$

$$d_2 = d_1 - \sqrt{\sigma_{\mathrm{IDI}}}. \tag{6.37}$$

The proof of this result is obtained directly from Proposition 6.4.

In case where we are dealing with a put option, the put-call parity provides the result. The inclusion of this proposition is motivated by the fact that from a practitioner's point of view, the result stated in Proposition 6.5 is very convenient.

7. Implied Market Expectations of Monetary Policy

In the USA, the Federal Reserve monetary policy has traditionally focused on controlling short-term interest rates through the Fed funds rate. Therefore, it follows that future short-term rates will be determined by future FOMC target ranges. Fed funds futures can be used to help infer the probabilities of future target ranges because their values are tied directly to the expected Fed funds rate. Moreover, these contracts are actively traded. In fact, two popular tools, the CME's FedWatch tool and Bloomberg's World Interest Rate Probability (WIRP) measure,[e] both use Fed funds futures to infer the implied probability of future FOMC decisions. Although these tools are useful, the methods only allow for two outcomes at each FOMC meeting and therefore restrict the size and direction of possible rate moves.

An alternative for third-party estimation of implied market interest rate probability is to recover directly the monetary authority decision probabilities, using market data on derivatives over interest rates. To do so, we used regularization techniques to deal with the ill-posedness nature of the problem as suggested by Müller (2009) and Zubelli (2005).

Let us suppose there are future contract quotes, $F_\star(t,T)$, representing the price at time t of an interest rate future with maturity T. By assuming that the futures contract resembles a zero-coupon bond we can use the result from Sec. 6 to price it and we can further calculate the pricing errors as $E(t,T) = F_\star(t,T) - F_M(t,T)$, where $F_M(t,T)$ represents the future price which incorporates monetary policy changes in scheduled meetings as derived in Eq. (6.7). Therefore, based on Müller (2009) we extract the implied risk-neutral probabilities, $\mathbb{Q}$, using the Tikhonov regularization method. Regularization techniques are heavily dependent on the existence of priors and we shall denote this prior as $\mathbb{Q}_t^\star$. Thus, we find $\mathbb{Q}$ that solves the following regularizated problem:

$$\min_{\mathbb{Q}_t} \ \frac{1}{2}\mathbb{Q}_t E(t,T)^T E(t,T)\mathbb{Q}_t + \gamma\|\mathbb{Q}_t - \mathbb{Q}_t^\star\|^2, \tag{7.1}$$

$$\text{s.t.} \begin{cases} \sum \mathbb{Q}_t(\theta_{\tau_{j+n}}) = 1, \quad \forall\, n, \\ \mathbb{Q}_t(\theta_{\tau_{j+n}}) \geq 0. \end{cases} \tag{7.2}$$

Here, we have used as prior $\mathbb{Q}_t^\star = \mathbb{Q}_{t-1}$ that is, the result for $t-1$ is used as *a prior* for the problem in t. As in Müller (2009) we have chosen $\gamma = 10^{-1}$ but unlike the author only futures contracts have been used mainly for two reasons: (i) futures are

[e]The authors thanks Bruno Dupire for bringing the existence of this index to our attention.

much more liquid than options, so including the later we could add more noise to the problem, (ii) our main goal in this paper is pricing options so we kept options to be used only for this purpose.

7.1. *Implied probabilities — USA*

Interest rate futures, specifically the 30-Day Fed fund futures contracts traded at CME Group, are not only useful tools for speculators and hedgers. These contracts offer valuable information that can be used to estimate the market's view of the probability of a rate change by the Federal Reserve.

Federal funds futures are one-month interest rate futures contracts, which are generally the difference between the average market rate during the delivery month and the futures rate at the time the contract was bought, multiplied by the notional amount, and they are settled in cash. Federal funds futures contracts are traded at CME Group and are quoted in terms of a price, which is calculated as 100 minus the realized average Fed funds rate for the delivery month.

Here, we show how to use our model to extract future decisions of the FOMC. In order to accomplish that, we consider quotes for 30-Day Fed fund futures ranging from 9 September 2017 to 12 December 2017. In early April, the general market expectation was for the Fed to start tightening the interest rate by June at the latest, given signs of continuing improvement in the economy.

The market's assessment of the future change by the Federal Reserve changed over the days preceding the meeting, but the most likely outcomes were stable and given by $\mathcal{A} = \{0\,\text{bps}, +25\,\text{bps}, +50\,\text{bps}\}$. Therefore, using market prices and $\mathcal{A}$ we are able to solve the regularizated problem (7.1) and extract the implied risk-neutral probabilities, $\mathbb{Q}(\theta_{\tau_j} = a_i)$ such that $a_i \in \mathcal{A}$:

The FOMC had a meeting scheduled on 13 December 2017, when they chose to increase the target range by 25 bps. We can see from Fig. 4 that market expectations about the outcome changed over the days before the meeting, but the market converged towards the FOMC decision.

7.2. *Implied probabilities — Brazil*

We also choose to extract the implied probabilities for future decisions of the monetary authority in Brazil for two reasons. First, there is a very liquid market for overnight interest rate futures in Brazil. Second, Brazil has adopted a inflation targeting regime since 1999 with scheduled meetings to define the target rate, and interest rate derivatives are used by market participants to bet on future monetary decisions.

The main interest rate derivatives in Brazil are DI futures, which are quoted in terms of rates and are traded in basis-points, but positions are recorded and tracked by the present value of the contract, called PU. Therefore each possible target rate will also have a market-based probability associated with it. The overnight interest

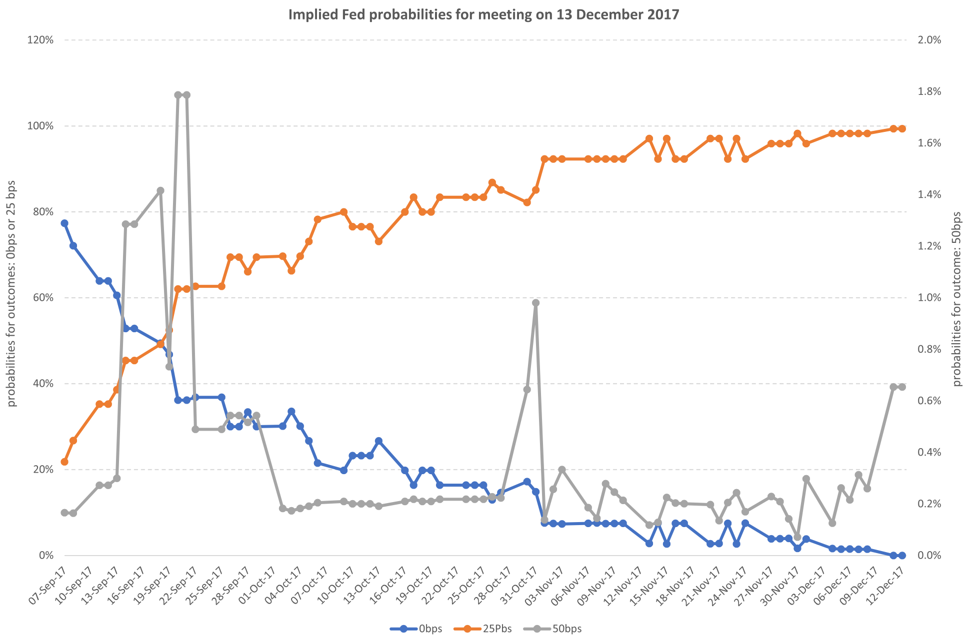

Fig. 4. Implied probabilities extracted from Fed Fund Futures for the Federal Reserve meeting on 13 December 2017. Tikhonov regularization method with $\gamma = 10^{-1}$. Numbers in left axis express a 25 bps interest rate hike (orange line) or Fed's interest rate maintenance (blue line). Right axis: 50 bps interest rate hike (gray line).

rate futures, DI futures,[f] traded at B3, is one of the most liquid short-term interest rate contracts in emerging markets, and the average volume of 2.3 million contracts traded daily is significant even for developed markets.

The notional value of the contract is BRL 100,000 (approximately USD 30,675 as of 3 February 2018). For a given day t the present value is obtained by discounting the notional value of the contract by the expected overnight interest rate from t up to the day prior to expiration, T. Therefore, at time t we can calculate the present value[g] (PU) of a DI-futures with expiration date of T as

$$PU_t = \mathbb{E}(e^{-\int_t^T r_s ds} \mid \mathcal{F}_t) \times 100{,}000. \tag{7.3}$$

From Eq. (7.3) we verify that the DI futures is very similar to a zero-coupon bond, except that it pays margin adjustments every day. The fact that the contract resembles a zero-coupon bond allows us to use the results derived at earlier sections to extract the implied market transition for $(\theta_t)_{t\geq 0}$ and use them for pricing options. Moreover, the number of possible target rate alternatives that are likely

[f] Ticker: DI1.
[g] In practice, the Brazilian convention for interest rate is exponential compound 252 business day (BD) and margin adjustment are calculated by the formula: $PU_t = 100{,}000/(1 + r_t)^{BD/252}$.

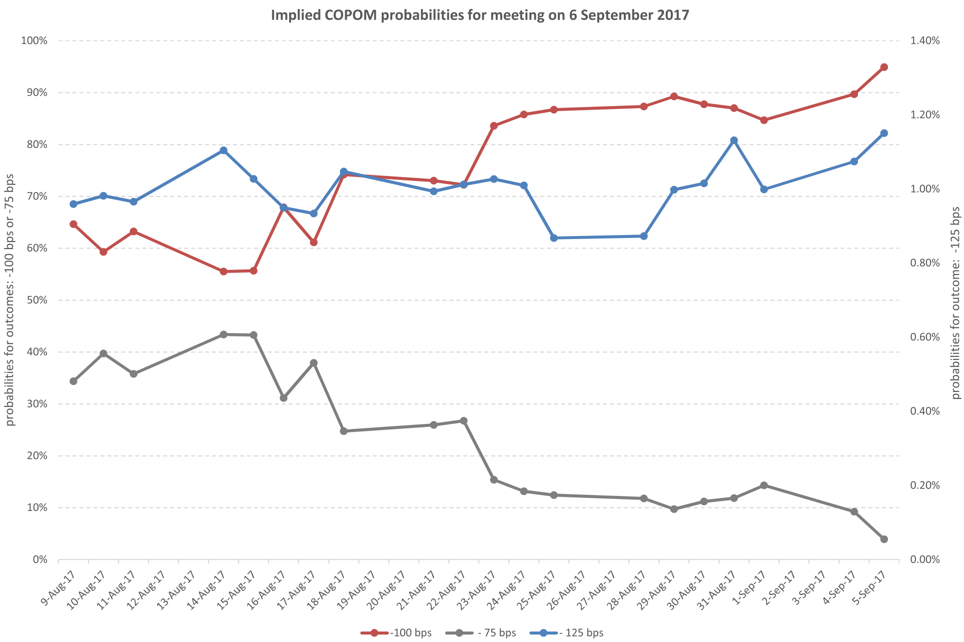

Fig. 5. Implied probabilities extracted from DI Futures for the COPOM meeting on 6 September 2017. Tikhonov regularization method with $\gamma = 10^{-1}$. Numbers in left axis refer to implied probabilities of 75 bps interest rate cut (gray line) or 100 bps interest rate cut (red line). Numbers in right axis represent implied probabilities of a 125 bps interest rate cut (blue line).

to be considered at any particular COPOM meeting is usually small, usually three or fewer. We assume that $\mathcal{A} = \{-125\,\text{bps}, -100\,\text{bps}, -75\,\text{bps}\}$ and we calibrate the model for every day from August 2017 to September 2017 to extract the market probabilities of the two next COPOM decisions by solving the optimization problem in Eq. (7.1).

Figure 5 shows implied probabilities of future outcomes for the monetary authority in Brazil extracted from DI1 futures prices on different dates. The shape of any given curve provides an indicator of market expectations on a particular day of future movements of interest rates. Day-to-day changes in market sentiment are reflected by shifts in the curves. The COPOM had a meeting scheduled for 6 September 2017, when the monetary authority chose to decrease the target rate by 100 bps.

Even though the results seem to be a valuable tool for extracting market expectations, this is not the paper's goal, we only resort to this exercise here to recover market expectations, implied by liquid instruments, for pricing less liquid instruments, such as options, in a consistent manner. For those interested in understanding how market implied expectations are linked to future changes in the interest rate, the paper by Nkwoma (2017) presents new evidence on the asymmetric effect of anticipated and unanticipated monetary policy changes.

8. Real Market Price Options

In this section, we present the results of our methodology applied to pricing interest rate options. The results are applied to two different types of short-term interest rate options: IDI options traded in Brazil and 30-Day Fed funds options traded at CME Group. For both sets of options we compare traded prices with the model derived in this paper with two competing models and show which one has a smaller pricing error. The competing models are Black's (1976) model and Merton's (1976) model.

In the following tables, we apply the model presented in Proposition 6.5 (Black DTJ) when the market expectation regarding future monetary policy decisions is extracted via the regularization methodology described in Sec. 7. The results obtained are compared with the prices of the IDI put options traded at B3 for a selected number of days. The results of the proposed model are also compared with Black's traditional model, widely used by practitioners in Brazil, as well as Merton's (1976) model that includes random jumps.

We present in Table 1 the results of the models, as well as market prices of different put options with maturity on 2 October 2017. For the selected days there was only one COPOM meeting before the option's maturity, scheduled for 6 September 2017. The volatility used in all the models was the same and estimated from the historical standard-deviation of the spot IDI index for an one-year window. Furthermore, for Merton's (1976) model it was assumed that the expected number of jumps was 8 per year (the total number of COPOM meetings expected in a year) while the jump amplitude was set at 10%.

For each model, the option price is presented, and since we do not have a traded price equal to zero we also present the percentage error. For all strikes the model that incorporates deterministic timed jumps (Black DTJ) exhibits a lower pricing error than any other model tested. It is interesting to observe that, as expected the inclusion of random jumps as in Merton's (1976) model reduces the pricing error

Table 1. IDI options — one scheduled meeting before the option's maturity on 2 October 2017.

Day	Strike	Prices in BRL				Percentage error		
		Trade	Black (76)	Black DTJ	Merton (76)	Black (76)	Black DTJ	Merton (76)
27 July 17	243.600	3.13	0.78	3.36	0.80	75.0%	−7.3%	74.6%
27 July 17	243.650	36.67	28.98	35.84	29.43	21.0%	2.2%	19.7%
27 July 17	243.700	81.70	77.97	84.36	79.18	4.6%	−3.3%	3.1%
28 July 17	243.600	3.22	0.89	3.31	0.90	72.4%	−3.0%	72.0%
28 July 17	243.650	37.34	30.19	35.84	30.65	19.1%	4.0%	17.9%
28 July 17	243.700	82.72	79.25	84.39	80.46	4.2%	−2.0%	2.7%
4 August 17	243.600	1.22	0.01	1.23	0.01	99.0%	−0.8%	99.0%
4 August 17	243.650	35.05	29.86	36.58	30.26	14.8%	−4.4%	13.7%
Mean percentange error (MPE)						38.8%	−1.8%	37.8%

when compared to Black's (1976) model in which the interest rate is a continuous process.

To assess the robustness of the results, we repeated the analysis using a different set of put options, the difference here comes from the fact that there were two COPOM meetings scheduled before the option's maturity. These options had expiration on 2 January 2018 and the COPOM meetings were scheduled to be held on 6 September 2017 and 25 October 2017. As can be seen in Table 2, the model proposed in this paper, which incorporates scheduled meetings, outperformed all other models tested for every day of our analysis.

Finally, we evaluated the quality of the model proposed in this paper for pricing 30-Day Fed funds options traded at CME and as in the case of IDI options, we made a comparison among models, we assessed the pricing errors of three different models: the model presented at Proposition 6.5 (Black DTJ), Black's (1976) model and Merton's (1976) model. The volatility used in all the models was the same and estimated from the historical standard-deviation of the effective federal funds rate (EFFR) for an one-year window. In addition, for Merton's (1976) model it was assumed that the expected number of jumps was also 8 per year (the total number of FOMC meetings expected in a year) while the jump amplitude was set at 5%. Unlike the IDI options, the 30-Day Fed funds options were very illiquid so we summarized our results in Table 3.

We see from Table 3 that the model developed in this paper, which includes deterministic-timed jumps (Black DTJ) also exhibited a lower pricing error than any other model tested. Black's model with deterministic timed jumps outperformed all other models, the performance was superior both for calls and puts and also for the case where we had more that one FOMC meeting before the options maturity.

Table 2. IDI options — two scheduled meetings before the option's maturity on 2 January 2018.

Day	Strike	Prices in BRL				Percentage error		
		Trade	Black (76)	Black DTJ	Merton (76)	Black (76)	Black DTJ	Merton (76)
8 September 17	247.900	20.35	0.18	27.49	0.18	99.1%	−35.1%	99.1%
8 September 17	248.000	90.89	73.28	96.03	74.95	19.4%	−5.6%	17.5%
8 September 17	248.100	179.12	171.05	175.87	174.95	4.5%	1.8%	2.3%
11 September 17	247.900	21.48	0.14	27.50	0.15	99.3%	−28.0%	99.3%
11 September 17	248.000	88.48	72.42	96.06	74.05	18.1%	−8.6%	16.3%
11 September 17	248.100	176.44	170.22	175.92	174.05	3.5%	0.3%	1.4%
18 September 17	247.900	27.03	2.85	26.75	2.91	89.4%	1.1%	89.2%
18 September 17	248.000	105.47	95.39	104.97	97.37	9.6%	0.5%	7.7%
18 September 17	248.100	197.83	193.35	201.07	197.37	2.3%	−1.6%	0.2%
19 September 17	247.900	24.81	0.73	23.61	0.74	97.1%	4.8%	97.0%
19 September 17	248.000	97.97	87.31	95.54	89.10	10.9%	2.5%	9.0%
19 September 17	248.100	189.34	185.30	189.24	189.10	2.1%	0.1%	0.1%
Mean percentange error (MPE)						29.1%	−0.1%	27.6%

Table 3. 30-Day fed funds options.

Day	Maturity	Type	# Meetings	Strike	Prices in USD				Percentage error		
					Trade	Black (76)	Black DTJ	Merton (76)	Black (76)	Black DTJ	Merton (76)
2 February 18	30 Mar 18	call	1	98.50	0.020	0.0110	0.0180	0.0130	45.0%	10.0%	35.0%
9 February 18	30 Mar 18	call	1	98.37	0.040	0.0205	0.0370	0.0305	48.8%	7.5%	23.8%
21 March 18	30 Mar 18	put	1	98.25	0.025	0.0180	0.0230	0.0200	28.0%	8.0%	20.0%
14 March 18	30 Jul 18	put	3	98.06	0.015	0.0100	0.0130	0.0120	33.3%	13.3%	20.0%
Mean percentange error (MPE)									38.8%	9.7%	26.3%

These findings corroborate our assumptions that the inclusion of scheduled events improves the pricing of overnight interest rate derivatives. Finally, we observe that our methodology is flexible enough for modeling equally well out-of-the money and in-the money options without any assumption over the implied volatility shape, i.e. smile or smirk. An important consequence of this framework is the fact that by construction options and ZCB will embed the same probabilities regarding the future monetary policy decisions.

A final remark on our framework concerns its comparison to Piazzesi (2005), which also models future monetary outcomes. While option prices in her framework are obtained by using numerical methods to solve a time dependent ordinary differential equation ours only incur in a couple of Black and Scholes-like valuations.

9. Conclusion

Many countries such as Brazil, England and the USA announce their monetary policy decisions at regularly scheduled meetings, and some have adopted inflation targeting as a strict rule for conducting their monetary policy. Under an inflation targeting framework, the central bank strives to meet a publicly announced inflation target using the monetary policy instruments at its disposal. Transparency and accountability of monetary policy are two important features of this framework. According to common practice among central banks that have adopted the inflation targeting regime, the target interest rate is set in their decision-making meetings according to a pre-announced schedule. In turn, market participants have carefully tracked all scheduled meetings where the target interest rate is set and traded derivatives to bet on possible outcomes.

Standard interest rate models are not suitable for handling deterministic timed events, and some level of mispricing is present when applying such models to pricing interest rate derivatives. Based on that, we developed in this paper a stochastic interest rate model able to endogenously incorporate monetary announcements. The model incorporates future monetary decisions and therefore allows pricing of both zero-coupon bonds and options in a consistent manner.

We apply our model for pricing options traded in Brazil and the USA. Brazil is a valid country to apply our model due to its adoption in 1999 of an inflation targeting regime, and also because it hosts a very liquid overnight interest rate derivatives market, which is used by participants to bet on future monetary decisions. We also apply our model to pricing 30-day Fed funds options traded at CME. When compared to market prices, the model with deterministic timed jumps outperformed all other models, the performance was superior both for calls and puts and also for the case where we had more that one scheduled meeting before the options maturity.

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