# Investment in Education and Strategic Asset Allocation

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**Abstract**

This work analyzes the optimal allocation of assets over the life cycle of an investor who has the option to invest in his education. We characterize the optimal spending rule for education, and compare the optimal asset allocation of such investors with the optimal portfolios of investors with no extra education. In particular, we are able to describe how the criteria for investing in risky assets change through time as investors acquire additional education. Our main findings are as follows. Investors undergoing the educational process tend to invest proportionally more in risky assets during that period than other investors. After the educational period investments in the risky asset usually decrease. In the working period after the investment in education a negative covariance between the working income and the returns of the risky asset induces the proportional investment of educated agents in the risky asset to be larger than the proportional investment of non-educated agents. For positive covariances, as it increases, the proportion of wealth invested in the risky asset decreases for both agents. Finally, in the retirement period the investments of people who invested in education earlier are not distinguishable from those of non-educated.

Keywords: Portfolio Choice; Saving; Education

JEL Codes: G11; E21; I22

# 1 Introduction

A common investment strategy suggested by financial advisors follows a simple rule of a linearly decreasing percentage of investors' wealth in a diversified portfolio of risky assets as one grows older (see Malkiel, 1996). The problem of asset allocation over the life cycle has received special attention from scholars and finance practitioners given its practical significance for each investor during different periods of life. Despite this importance, there is no consensus about the optimal portfolio choice.

Since the earliest work in this area, the theoretical predictions have differed from the observed empirical patterns. On the one hand, several empirical papers find evidence for a hump-shaped form of risky asset holdings over the life cycle (e.g. Ameriks and Zeldes, 2004, Faig and Shum, 2002 and Shum, 2006), with younger people holding a low share of risky assets. On the other hand, the original theoretical models tell us that younger people should hold a large fraction of their wealth in risky assets and that this proportion should decrease as the agent grows older, as stated by Merton (1969) and Samuelson (1969). Merton (1971) has shown this holds when deterministic working income is added to the investment endowment in a complete-market setting. Such results are consistent with the classical mean-variance model of portfolio choice, according to which the holdings of aggressive and conservative investors should differ only in their holdings of cash; the relative proportion of long-term bonds and stocks should be the same for both types of investors, regardless of the investors' planning horizon. As pointed out by Canner, Mankiw, and Weil (1997), this classical approach is inconsistent with the conventional financial advice in the markets.

Seeking to understand the empirical facts, several models emerged complementing Merton's (1969) and Samuelson's (1969) seminal models. Viceira (2001) added complexity to the theoretical setting by allowing working income to be uncertain, showing that the investment horizon may affect the optimal portfolio allocation. The introduction of uncertain sources of income was indeed the crucial ingredient to rethink strategic asset allocation. Realizing the failure of Samuelson's and Merton's analysis, some authors suggested alternative techniques. By analyzing who should buy long-term bonds through a multi-period model incorporating uncertainty in fixed-income investment, Campbell and Viceira (2001) have shown that aggressive long-term investors should hold essentially stock, whereas conservative long-term investors should hold more long-term bonds together with some small amount of cash. The intuition for this result is that, for long-term investors, the riskless asset is not cash, but rather long-term bonds.

Cocco, Gomes, and Maenhout (2005) and Campbell, Cocco, Gomes, and Maenhout (2005) characterize the solution of a more complex problem in this same conceptual framework, namely the optimal choice of portfolio when considering consumption over a life cycle and simultaneous investment in retirement wealth. Their results from calibration show that the low holdings of the risky assets by younger people could be explained only by either a significant shock in labor income or an extremely high correlation between labor income and returns of the risky assets.

In contrast to these papers, we analyze how the optimal allocation of assets over the life cycle of an investor may be affected by the alternative to invest in an education program. One such educational program is modeled so as to increase the future expected income by assuming the possibility of promotion in the job market. Our model considers basic tradeoffs in this decision process. First, to invest in education an agent will give up some current consumption and reduce the investment in financial assets; second, the expected increase in future income must compensate the reduced future investment horizon; third, the change in the optimal risky investment will depend on the correlation between the returns of the risky assets and the future labor income.

Unlike the findings reported in most earlier theoretical papers, but in line with the empirical evidence, our model predicts that for a reasonable level of risk aversion, a hump-shaped portfolio strategy over the life cycle is optimal. To be precise, we find that investors who seek higher education: 1) tend to invest proportionally more in risky assets during the educational period than do other investors; 2) the investment in risky assets increases as the expected time until promotion diminishes; 3) following the educational period, investments in risky assets usually decrease. Furthermore, in the period of active life there are three possibilities: if the work income covaries positively (and at a high enough level) with the return of the risky asset, investments in that risky asset are proportionally less than the investments of non-educated individuals, since the working income is considered to be a risky source; if the covariance is negative, the working income can be seen as an insurance and the investments in risky assets are proportionally greater among educated investors; if the covariance is zero, the investments are proportionally equal to those of non-educated individuals in the period following promotion and proportionally greater during the education period.

This work is organized as follows. In Section two we extend Viceira's (1998) two-period model to incorporate the possibility of investing in self education, allowing for future job promotion with corresponding income increase. Section three is concerned with the development of a similar model, now considering an investor living for an arbitrarily large number of periods. In Section four we provide different simulations for this dynamic allocation model, considering the life cycle of investors who have improved their education. Section five concludes.

# 2 Two-Period Theoretical Model

In this section we develop a two-period model where an economic agent (hereinafter named the investor) maximizes his or her utility allocating optimally his or her wealth at time $t$ between the available risky assets and a risk-free asset. When making this decision, the investor must also decide how much to invest in his or her education. The underlying idea is that an education upgrade may possibly increase his or her future earnings in the second period.

In this model the agent's education can be seen as an investment, leading to a promotion and a consequent higher wage[1]. In the first period the investor supports the costs of his or her education, whereas in the second period he or she may have income compensation due to the promotion derived from the educational upgrade. The education can also provide “status benefit” for the investor[2].

The investor receives income from his or her work at times $t$ and $t+1$, denoted respectively by $L_t$ and $L_{t+1}$. If investment in educational upgrade is made at $t$, he or she receives a raise in the $t+1$ income. Notice that this model has only one investment period. Also, the investor is assumed to have an initial endowment $W_t$ before receiving the income $L_t$.

We assume that the investor makes decisions related to portfolio and education choice at time $t$. At $t+1$ the agent consumes the amount returned by the portfolio plus the income generated by his or her work at that point in time.

## 2.1 Certain Promotion

Consider an investor with a two-period life. Following Campbell and Viceira (2002), and discounting the current consumption of this agent, the problem of the investor is reduced to know how to allocate today’s wealth, depending on two factors: future consumption and current expenditure with education. The utility function is assumed to be given by

$$\beta \frac{C_{t+1}^{1-\gamma}}{1-\gamma} + v(X_t)$$

where $C_{t+1}$ is the $t+1$ consumption, $X_t$ is the expenditure with education at time $t$, the factor $\gamma$ denotes the risk aversion coefficient, the discount factor is denoted by $\beta$ satisfying $0 < \beta < 1$, and the function $v$ characterizes the utility of increased status originating with the expenditure in

[1]See Arcidiacono (2004), Mishler (1982), and Clark and Anderson (1992).
[2]See Mishler (1982) and Clark and Anderson (1992).

education (such that $v' > 0$ and $v'' < 0$). Notice that the investment $X_t$ may increase utility in two ways. First, by generating more future income, thus increasing future consumption $C_{t+1}$; second, by increasing the status of the agents who invest in education upgrade.

We next discuss the flow of working income. The working income at time $t+1$ is assumed to be uncertain and described by the random variable $L_{t+1}$. We assume furthermore that the conditional distribution of the working income is log-normal. Defining $l_{t+1} \equiv \log(L_{t+1})$, this implies that

$$l_{t+1} \sim N(l, \sigma_l^2).$$

The economy is also characterized by the assets that are transacted. We consider that there are two types of assets in the market, namely a risky asset with random return $R_{t+1}$, and a riskless asset with return $R_f$. In this model we assume that the logarithm of the excess return $R_{t+1} - R_f$ is a log-normal variable. Defining $r_f \equiv \log R_f$ and $r_{t+1} \equiv \log R_{t+1}$, this implies that

$$r_{t+1} - r_f = \mu + u_{t+1}$$

where $\mu \equiv E_t[r_{t+1} - r_f]$, and the unexpected log-return of the risky asset is conditionally Normal. In other words,

$$u_{t+1} \sim N(0, \sigma_u^2).$$

Additionally, we assume a possible correlation between the returns of the risky asset and the working income, given by

$$\text{cov}_t(l_{t+1}, r_{t+1}) = \sigma_{lu}.$$

The return on the agent's portfolio is thus given by

$$R_{pt+1} = \alpha_t R_{t+1} + (1 - \alpha_t) R_f.$$

For simplicity, we shall work with log-returns. Denoting by $r$ the logarithm of the return $R$, a second-order Taylor expansion leads to

$$r_{pt+1} = r_f + \alpha_t(r_{t+1} - r_f) + \frac{1}{2}\alpha_t(1-\alpha_t)\sigma_u^2.$$

Finally, consider the case where an investment is made in upgrading education. Let $T_{t+1}$ denote the new income of the investor at time $t+1$, after the promotion due to the upgrading of his or her education. For consistency, we shall denote by $\tau_{t+1} = \log(T_{t+1})$ the logarithm of this new income. In that case, we assume that the investor's log-wage $l_{t+1}$is increased at time $t+1$ by a multiplicative factor $\bar{z} > 1$, leading to a working log-income $\tau_{t+1} = \bar{z}l_{t+1}$, following, therefore, a stochastic process with log-normal conditional distribution

$$\tau_{t+1} \sim N(\tau, \sigma_\tau^2).$$

Although the income increase is stochastic, this increase is perfectly well-defined once the realization of $l_{t+1}$is known. For this reason we shall refer to this situation as certain increase, as opposed to the case in the next section, where $\tau_{t+1}$ will depend of the expenditure $X_t$ made in education. We also assume the possibility of some correlation between the wage increase and the return of the risky asset,

$$\mathrm{cov}_t(\tau_{t+1}, r_{t+1}) = \bar{z}\mathrm{cov}_t(l_{t+1}, r_{t+1}) = \bar{z}\sigma_{lu} \equiv \sigma_{\tau u}.$$

### 2.1.1 The Optimization Problem

From the available wealth, the investor will spend $X_t$ with education, will invest a fraction $\alpha_t$ of the remaining wealth in the risky asset, and a fraction $1-\alpha_t$ in the riskless asset. The optimization problem is to choose $\alpha_t$ and $X_t$ in order to maximize the expected utility

$$\max_{X_t,\alpha_t} E_t\left[\beta\frac{C_{t+1}^{1-\gamma}}{1-\gamma}+v(X_t)\right] \tag{1}$$

subject to

$$\begin{aligned} C_{t+1} &= (W_t+L_t-X_t)(1+R_{pt+1})+T_{t+1} \\ R_{pt+1} &= \alpha_t R_{t+1}+(1-\alpha_t)R_f. \end{aligned} \tag{2}$$

Introducing both restrictions in the objective function, the problem becomes to maximize in order to $X_t$ and $\alpha_t$ the following expression

$$E_t\left[\beta\frac{\{(W_t+L_t-X_t)\left[1+R_f+\alpha_t(R_{t+1}-R_f)\right]+T_{t+1}\}^{1-\gamma}}{1-\gamma}+v(X_t)\right].$$

The first-order conditions to solve this problem are respectively,

$$\begin{aligned} \frac{v'(X_t)}{u'(C_{t+1})} &= \beta(1+R_{pt+1}) \\ E_t[C_{t+1}^{-\gamma}(1+R_{t+1})] &= E_t[C_{t+1}^{-\gamma}(1+R_f)]. \end{aligned}$$

Taking the logarithm in the last equality and rewriting terms, we obtain

$$E_t[r_{t+1}-r_f]+\frac{\sigma_u^2}{2}=\gamma\text{cov}_t(c_{t+1},r_{t+1}). \tag{3}$$

Now let $I_t \equiv W_t+L_t-X_t$. Dividing expression (2) for the budget constraint by $L_{t+1}$, it follows that

$$\frac{C_{t+1}}{T_{t+1}} = \frac{I_t}{T_{t+1}}(1 + R_{pt+1}) + 1.$$

Following Campbell (1993), we take logarithms on both sides to obtain

$$c_{t+1} - \tau_{t+1} = \log\left\{\exp\left[(i_t - \tau_{t+1}) + r_{pt+1}\right] + 1\right\}.$$

Expanding the right-hand side around the expected values of $r_p$ and $i - \tau$, and defining $\theta \equiv \{(i - \tau) + r_p\}$, we have

$$c_{t+1} - \tau_{t+1} \approx k + \rho(i_t - \tau_{t+1}) + \rho(r_{pt+1}),$$

implying that

$$c_{t+1} \approx k + \rho(i_t + r_{pt+1}) + (1 - \rho)\tau_{t+1},$$

where

$$\rho = \frac{\exp[(i - \tau) + r_p]}{1 + \exp\theta};\ 0 < \rho < 1,$$

and $k = \log\left[\exp\left(\theta\right) + 1\right] - \rho\left(i + r_p - \tau\right) > 0$. Replacing the expression for $c_{t+1}$ in (3) we have

$$E_t[r_{t+1} - r_f] + \frac{\sigma_u^2}{2} = \gamma\rho\alpha_t\sigma_u^2 + \gamma(1 - \rho)\bar{z}\sigma_{lu},$$

and the optimal fraction invested in the risky asset is thus given by

$$\alpha_t = \frac{1}{\rho}\left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) - \frac{(1 - \rho)}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right) - \frac{(1 - \rho)(\bar{z} - 1)}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right).$$

### 2.1.2 Main results and interpretation

The three fundamental equations of this model are

$$\frac{v'(X_t)}{u'(C_{t+1})} = \beta(1 + R_{pt+1}) \tag{4}$$

$$c_{t+1} \approx k + \rho(i_t + r_{pt+1}) + (1-\rho)\tau_{t+1} \tag{5}$$

$$\alpha_t = \frac{1}{\rho}\left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) - \frac{(1-\rho)}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right) - \frac{(1-\rho)(\bar{z}-1)}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right) \tag{6}$$

First, equation (4) expresses the marginal rate of substitution between consumption at time $t+1$ and the amount spent with education at time $t$. This rate is a function of two variables: intertemporal discount ($\beta$) and the return of portfolio ($R_{pt+1}$). Notice that investors with higher $\beta$ will have a higher marginal rate of substitution, other things equal, spending less with their education today and/or consuming more in the future. Also, an increase in the expected portfolio return, other things equal, will lead the investors to spend less with their education today and/or expect to consume more in the future.

Equation (5) describes the optimal consumption as given by a constant $k$ plus a weighted sum of the net wealth $i_t$, working income together with education's gains $\tau_{t+1}$, and portfolio's return $r_{pt+1}$. All three of these variables are positively correlated with investor's consumption at $t+1$. The weighting parameter $\rho$ is important to determine the optimal portfolio choice.

Finally, equation (6) describes the optimal allocation of wealth in the risky asset. We examine the three components separately. The first component describes the optimal wealth allocation when the working income is idiosyncratic, *i.e.*, not correlated with the return of the risky asset. The second component reflects a hedging strategy to cover the uncertainty associated to the working income. In other words, the utility for holding a risky asset is not only driven by the excess return with respect to its variance, but also from the assets' ability to protect future consumption from fluctuations in the working income. If the covariance between $l_{t+1}$ and $r_{t+1}$ is positive, then the working income may be considered as a risky asset, allowing the proportion of wealth invested in the financial risky asset to decrease. If that covariance is negative, however, this means that the risky asset is a good protection against the uncertainty of the working income, leading to a larger proportion of wealth invested in the financial risky asset. The third factor is also a hedging component. The initial spending with education can also be seen as an investment, generating a future payoff to the investor. The rise in wage will be a function of the future wealth and may be correlated with the return of the financial risky asset. In this sense, and if such correlation is negative, the investment in the financial risky asset may also be seen as an insurance. On the other hand, if the correlation is positive, the gain may be seen as a risky asset and the investor reduces his or her initial investment in the financial risky asset.

## 2.2 Uncertain Promotion

The structure of this section is very similar to the previous. The main difference is that here we consider that the income increase will depend on the amount $X_t$ spent in education. In other words, to spend in education does not ensure that the promotion will be obtained, and that future income $l$ will be multiplied by $\bar{z}$. We assume hereby that $z$ is an increasing, monotonic function depending on $X_t$, such that $z = 1$ whenever $X_t = 0$.

### 2.2.1 The New Optimization Problem

We consider in this section the existence of an increasing, differentiable function

$$z : X \to [1, \bar{z}], \text{ with } z' > 0,$$

such that the amount received at $t+1$ after investing in education is now $L_{t+1}^{z(X_t)}$ instead of $L_{t+1}$. The investor must then maximize the utility given by equation (1), subject to the constraints

$$\begin{aligned} C_{t+1} &= (W_t + L_t - X_t)(1 + R_{pt+1}) + L_{t+1}^{z(X_t)} \qquad (7) \\ R_{pt+1} &= \alpha_t R_{t+1} + (1-\alpha_t) R_f. \end{aligned}$$

Notice that the main difference with respect to the previous section is that restriction (7) incorporates the function $z(X_t)$. As before, we substitute both constraints in the expression for $C_{t+1}$ in the argument of the utility, and transform the problem into the maximization of

$$E_t \left[ \beta \frac{\left\{ (W_t + L_t - X_t) \left[ 1 + R_f + \alpha_t (R_{t+1} - R_f) \right] + L_{t+1}^{z(X_t)} \right\}^{1-\gamma}}{1-\gamma} + v(X_t) \right].$$

The first-order conditions for this problem are

$$\begin{aligned} \frac{v'(X_t)}{u'(C_{t+1})} &= \beta \left( 1 + R_{pt+1} - z'(X_t) L_{t+1}^{z(X_t)} \right) \\ E_t[C_{t+1}^{-\gamma}(1 + R_{t+1})] &= E_t[C_{t+1}^{-\gamma}(1 + R_f)]. \end{aligned}$$

We can now manipulate these equations, just as in the previous section, to isolate the optimal investment in the risky asset and the optimal future consumption. We thus have

$$c_{t+1} \approx k + \rho(i_t) + \rho(r_{pt+1}) + (1-\rho)\, \tau_{t+1}$$

where now

$$\rho = \frac{\exp\left(i - \tau\right)}{1 + \exp\theta} \qquad (8)$$

with $\theta = \{[i - z(X)l] + r_p\}$. Using the expression for $c_{t+1}$ we then obtain for the fraction of wealth invested in the risky asset the same formal expression as before

$$\alpha_t = \frac{1}{\rho}\left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) - \frac{(1-\rho)}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right) - \frac{(1-\rho)\left[z(X) - 1\right]}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right).$$

Notice, however, that there are several dependencies of this expression on the function $z(\cdot)$ through the presence of $\theta$ in the parameters $\rho_{r_p}$, $\rho_i$, and $\rho_\tau$, and an explicit dependence on the parameter $\rho_\tau$.

#### 2.2.2 Interpretation of the new solution

We start by analyzing the equations characterizing the solution of this problem. The first equation

$$\frac{v'(X_t)}{u'(C_{t+1})} = \beta\left[1 + R_{pt+1} - z'(X_t)L_{t+1}^{z(X_t)}\right] \tag{9}$$

reflects the marginal rate of substitution between the investor's consumption at time $t+1$ and the amount spent with his education at time $t$. The main difference with the previous case is the presence of the last term, related to the increase of future wage. Also notice that, since $z' > 0$, we have from equation (9) that

$$\frac{v'(X_t)}{u'(C_{t+1})} < \beta(1 + R_{pt+1}).$$

Hence, the marginal rate of substitution for the case of uncertain wage increase is lower than in the case of certain wage increase. Therefore, in this case of uncertain increase, other things constant, the investor spends more with education and/or the future consumption is less than in the case of certain increase. Uncertainty in wage increase provides an incentive to spend more with education today, increasing the proportion of expected future income generated by education.

The equation for optimal consumption

$$c_{t+1} \approx k + \rho(i_t) + \rho(r_{pt+1}) + (1-\rho)\,\tau_{t+1}$$

is formally similar to equation (5). Therefore, the intuition is the same.

Finally, the same formal similarity applies to the equation for the optimal fraction of wealth invested in the risky asset

$$\alpha_t = \frac{1}{\rho}\left(\frac{\mu + \frac{\sigma_u^2}{2}}{\gamma\sigma_u^2}\right) - \frac{(1-\rho)}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right) - \frac{(1-\rho)\,[z(X)-1]}{\rho}\left(\frac{\sigma_{lu}}{\sigma_u^2}\right). \tag{10}$$

The main difference with the previous section is that the investor now chooses the optimal portfolio taking into account the impact of how much is spent with education. Equation (10) reflects this effect in two ways. First, through the explicit presence of $z(X)$ in the last term. Second, because $\rho$ depends on $z(\cdot)$. In fact, from equation (8), $\rho$ is an increasing function of $z(\cdot)$ and therefore an increasing function of $X_t$.

So, if $X_t < z^{-1}(\bar{z})$, the first term in (10) is greater than in the case of certain wage increase, contributing to increase the proportion of wealth in the risky asset. However, notice that as $z(X) \to \bar{z}$, the case of certain wealth increase is recovered.

The second term clearly decreases with $X_t$, compensating at least partially the effect of the first term.

The third term is ambiguous with respect to the effect of an increase in $X_t$. In fact, the direct effect of $[z(X)-1]$ points to an increase of that term, but multiplication by $(1-\rho)/\rho$ indicates that the term may decrease as $X_t$ grows. However, a certain number of conclusions may be stated. Notice that as $X_t$ approaches $z^{-1}(\bar{z})$, the whole term tends to the value obtained in the case of a certain wage; also, as $X_t \to 0$, the term vanishes, since $z(X) \to 1$, and hence, would no longer influence the optimal asset allocation; finally, this points to the fact that, for small enough $X_t$, the whole third term increases with the amount invested in education.

# 3 Multiperiod Model

In this section we consider that an agent has a life with infinite periods of time. However, at each point in time there are two possible states of nature: either the agent is employed or retired. The agent is employed with probability $\pi^e$, in which case receives a working income, and is retired with probability $\pi^r = 1 - \pi^e$. This retirement probability shortens the expected horizon of the employment duration to $1/\pi^r$ periods. Moreover, the state of retirement is irreversible, implying that if an agent is retired from the job market, his or her income will be zero thereafter. After retirement, there is a constant probability of death $\pi^d$ at each point in time that makes the expected lifetime after retiring equal to $1/\pi^d$ periods. Similarly, if the investor has not retired, there is a probability $\pi^p$, at each point in time, that the investor is promoted. This promotion probability makes the expected education's time horizon to be $1/\pi^p$ periods. That state is also irreversible, in the sense that an investor who is promoted will spend no more money with his or her education. In the context of our model, this means that a promotion raises income in two ways. First, after the investor's promotion, the amount spent with education will be zero from then on ($X_t = 0$) and second, there will be a wage increase. Of course, at each point in time there is the probability $\pi^s = 1 - \pi^p$ that the agent does not get the promotion for one more period, in which case the investor keeps spending with his or her education ($X_t > 0$) and receives no wage increase.

Assuming that an agent is working for several periods, his working income is subject to a number of shocks. We model the working income as the process

$$L_{t+1} = L_t \exp\{g + \xi_{t+1}\}$$

where

$$\xi_{t+1} \backsim N(0, \sigma_\xi^2)$$

is a random variable whose realization is independent of the realization of the working income. The logarithm of this income process can also be characterized as

$$\Delta l_{t+1} = g + \xi_{t+1}.$$

The gains from promotion referred to above will be modeled in a similar way to the two-period model. In particular, we shall assume that when the promotion occurs, the investor receives a new income $\tau_{t+1}$ each period, where $\tau_{t+1} = z l_{t+1}$, with fixed $z > 1$. This leads to a logarithmic increase per period of the income after promotion, denoted by

$$\Delta\tau_{t+1} = z\Delta l_{t+1}.$$

The set of possible investments is characterized by two financial assets as before, a risky and a riskless asset, and the expected log-return in excess is given by $E_t(r_{t+1} - r_t) = \mu$. The unexpected log-return of the risky asset, denoted by $u_{t+1}$, is conditionally homoscedastic and serially non-correlated, although it may be contemporaneously correlated with innovations in the changes of the working log-income and, therefore, with the changes of the logarithm of the new income. We thus assume that

$$\begin{aligned}
var_t(u_{t+1}) &= \sigma_u^2 \\
cov_t(r_{t+1}, \xi_{t+1}) &= \sigma_{u\xi} \\
cov_t(r_{t+1}, \Delta l_{t+1}) &= \sigma_{u\Delta l} \\
cov_t(r_{t+1}, \Delta\tau_{t+1}) &= \sigma_{u\Delta\tau}.
\end{aligned}$$

Finally, we also assume that innovations in the risky asset do not depend on whether the investor is employed or retired. In other words, the state of nature defining whether or not income is received is completely idiosyncratic, not depending on the business cycle.

Consider an investor that consumes and invests at different points in time. At time $t$, his or her preferences with respect to consumption and investment $i$ periods later, are characterized by the utility function

$$U = \frac{C_{t+i}^{1-\gamma}}{1-\gamma} + v(X_{t+i}).$$

We assume, whenever necessary, that $v$ is CRRA, with the same risk aversion coefficient as the consumption term. As before, $\beta$ denotes the intertemporal discount factor with $0 < \beta < 1$.

## 3.1 The Optimization Problem

The investor faces the following intertemporal optimization problem.

$$\max_{\{C_{t+i}, X_{t+i}, \alpha_{t+i}\}_{i=0}^{\infty}} E_{t+i} \left\{ \sum_{i=0}^{\infty} \beta^i \left[ \frac{C_{t+i}^{1-\gamma}}{1-\gamma} + v(X_{t+i}) \right] | L_t, u_t, \xi_t \right\}$$

subject to

$$W_{t+i+1} = \left[W_{t+i} + \iota_e(1-\iota_p)L_{t+i} + \iota_e\iota_p T_{t+i} - (1-\iota_p)X_{t+i} - C_{t+i}\right](1 + R_{p,t+i+1}) \quad (11)$$

$$R_{p,t+i+1} = \alpha_t(R_{t+i+1} - R_f) + R_f, \quad (12)$$

where

$$\iota_e = \begin{cases} 1 & \text{when investor is employed} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\iota_p = \begin{cases} 1 & \text{when the agent is promoted} \\ 0 & \text{otherwise} \end{cases}$$ and the variable $W_{t+i+1}$ in equation (11) reflects the financial wealth, defined as the amount retained by the investor at the beginning of period $t+1$ in financial assets that were bought at time $t$.

In equation (12) the factor $\alpha_{t+i}$ denotes the proportion of wealth invested in risky assets. We also have $R_{t+i+1} = \exp(r_{t+i+1}), R_f = \exp(r_f)$. Three possibilities arise from the budget constraint in equation (11).

1. The investor may be employed and spend money with his own education, in which case

$$W^{es}_{t+i+1} = (W_{t+i} + L_{t+i} - X_{t+i} - C_{t+i})(1 + R_{p,t+i+1}). \tag{13}$$

2. An employed investor is promoted, and the restriction reads

$$W^{ep}_{t+i+1} = (W_{t+i} + T_{t+i} - C_{t+i})(1 + R_{p,t+i+1}). \tag{14}$$

3. The investor may be retired

$$W^{r}_{t+i+1} = (W_{t+i} - C_{t+i})(1 + R_{p,t+i+1}). \tag{15}$$

Depending on the case we are dealing with, there will be three different Euler equations for the optimization problem.

In the first case, we write Bellman's equation as

$$V(W_{t+i}) = \max_{\{C_{+it}, X_{t+i}, \alpha_{t+i}\}_{i=0}^{\infty}} \left[ \sum_{i=0}^{\infty} \beta^i \left( \frac{C_{t+i}^{1-\gamma}}{1-\gamma} + v(X_{t+i}) \right) \mid L_t, u_t, \xi_t \right]$$

corresponding to maximizing with respect to $C_{t+i}, X_{t+i}, \alpha_{t+i}$ the expression

$$\frac{{C^{es}_{t+i}}^{1-\gamma}}{1-\gamma}+v(X_{t+i})+\beta E_{t+i}\left\{\pi^e\left[\pi^p V(W^{ep}_{t+i+1})+(1-\pi^p)V(W^{es}_{t+i+1})\right]+(1-\pi^e)(1-\pi^d)V(W^r_{t+i+1})\right\}.$$

The first-order conditions for this problem are the following. For $\alpha_t$,

$$E_{t+i}[V'(W_{t+i+1})((1+R_{t+i+1})-(1+R_f))]=0$$

implying that

$$E_{t+i}\left[u'\left(C^{es}_{t+i+1}\right)(1+R_{t+i+1})\right]=E_t\left[u'\left(C^{es}_{t+i+1}\right)(1+R_f)\right].$$

With respect to $C$, we have

$$u'(C^{es}_{t+i})=\beta E_{t+i}\left\{\left[\pi^e\left\langle V'\right\rangle+(1-\pi^e)(1-\pi^d)\beta V'\left(W^r_{t+i+1}\right)\right](1-R_{p,t+i+1})\right\}$$

where

$$\left\langle V'\right\rangle=\pi^p V'\left(W^{ep}_{t+i+1}\right)+\pi^s V'\left(W^{es}_{t+i+1}\right)$$

and $V'(W)=u'(C)$ by the envelope theorem. This leads to

$$E_{t+i}\left\{\beta\left[\pi^e\left\langle\mathcal{C}\right\rangle+(1-\pi^e)\beta^r\left(\frac{C^r_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma}\right](1-R_{p,t+i+1})\right\}=1$$

where $\left\langle\mathcal{C}\right\rangle=\pi^p\left(\frac{C^{ep}_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma}+(1-\pi^p)\left(\frac{C^{es}_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma}$.

Finally, with respect to $X$, we have

$$v'(X^{es}_{t+i}) = \beta E_{t+i}\left\{[\pi^e\pi^s V'(W^{es}_{t+i+1})](1 - R_{p,t+i+1})\right\} \tag{16}$$

and $V'(W) = v'(X)$ by the envelope theorem. This leads to

$$\frac{1}{\beta} = E_{t+i}\left[\pi^e\pi^s \frac{v'(X_{t+i+1})}{v'(X_{t+i})}(1 - R_{i,t+i+1})\right].$$

In the second case (promoted investor), we have

$$V(W_{t+i}) = \max_{\{C_{t+i}, X_{t+i}, \alpha_{t+I}\}_{i=0}^{\infty}} \left[\sum_{t=0}^{\infty} \beta^i \left(\frac{C^{1-\gamma}_{t+i}}{1-\gamma} + v(X_{t+i})\right) \mid L_t, u_t, \xi_t\right]$$

corresponding to

$$\max_{\{C_{t+i}, X_{t+i}, \alpha_{t+i}\}_{t=0}^{\infty}} \frac{{C^{ep}_{t+i}}^{1-\gamma}}{1-\gamma} + v(X_{t+i}) + \beta E_{t+i}\left[\pi^e V\left(W^{ep}_{t+i+1}\right) + (1-\pi^e)\left(1-\pi^d\right) V\left(W^r_{t+i+1}\right)\right]$$

The first-order conditions for this problem are the following. For $\alpha_t$,

$$E_{t+i}[V'(W_{t+i+1})((1 + R_{t+i+1}) - (1 + R_f))] = 0$$

leading to

$$E_{t+i}[u'(C^{ep}_{t+i+1})(1 + R_{t+i+1})] = E_{t+i}[u'(C^{ep}_{t+i+1})(1 + R_f)].$$

With respect to $C$,

$$u'(C^{ep}_{t+i}) = [(\pi^e\beta E_{t+i}V'(W_{t+i+1}) + (1-\pi^e)(1-\pi^d)\beta E_{t+i}V'(W_{t+i+1}))(1 + R_{p,t+i+1})].$$

Let $\beta^r = \beta(1-\pi^d)$. Since $V'(W_{t+i}) = u'(C_{t+i})$, the first-order condition reads

$$E_{t+i}\left\{\left[\pi^e\beta\left(\frac{C^{ep}_{t+i+1}}{C^{ep}_{t+i}}\right)^{1-\gamma} + (1-\pi^e)\beta^r\left(\frac{C^r_{t+i}}{C^{ep}_{t+i}}\right)\right](1+R_{i,t+i+1})\right\} = 1. \tag{17}$$

Finally, in the third case (retired investor) we have

$$\begin{aligned} V(W_{t+i}) &= \max_{\{C_{t+i},X_{t+i},\alpha_{t+i}\}_{t=0}^{\infty}} \left[\sum_{i=0}^{\infty}\beta^i\left(\frac{C^{1-\gamma}_{t+i}}{1-\gamma} + v(X_t)\right)/L_{t+i}, u_{t+i}, \xi_{t+i}\right] \\ &= \max_{\{C_{t+i},X_{t+i},\alpha_{t+i}\}_{t=0}^{\infty}} \frac{C^{r\,1-\gamma}_{t+i}}{1-\gamma} + v(X_{t+i}) + \beta(1-\pi^d)E_{t+i}V(W^r_{t+i+1}). \end{aligned}$$

The first-order conditions are now

$$\begin{aligned} C &: \quad u'(C^r_{t+i}) = \beta(1-\pi^d)E_{t+i}V'(W_{t+i+1})(1+R_{p,t+i+1}) \\ V'(W) &= \quad u'(C) \\ &\Longrightarrow \quad u'(C^r_{t+i}) = \beta(1-\pi^d)E_{t+i}\{u'(C^r_{t+i+1})(1+R_{p,t+i+1})\} \\ \alpha_t &: \quad E_{t+i}[V'(W^r_{t+i+1})((1+R_{t+i+1}) - (1+R_f))] = 0 \\ &\Longrightarrow \quad 1 = E_{t+i}\left[\beta^r\left(\frac{C^r_{t+i+1}}{C^r_{t+i}}\right)^{-\gamma}(1+R_{i,t+i+1})\right]. \end{aligned} \tag{18}$$

Notice that $\beta^r = \beta(1-\pi^d)$ may be read as an effective discount rate, incorporating the probability of death. The effect of increasing the expected life after retirement (lowering $\pi^d$) is equivalent to having a larger effective discount factor $\beta^r$, leading the investor to give a relatively larger value to the present. This rephrases the same idea as in Viceira (2001).

## 3.2 An approximate log-linear solution

Just as in Viceira (2001), we follow Campbell and Viceira (1999) to find an approximate solution to the problems above, through the method of undetermined coefficients. This process requires three steps. First, we make a log-linear approximation of the budget constraints and of the Euler equation around the steady states. Second, we look for optimal policy functions for consumption and portfolio allocation using the log-linearized equations. Third, we identify the coefficients of the policy functions using the undetermined coefficient method.

This procedure is possible since the assumptions about preferences, working income, wealth transfer, and the set of investment possibilities ensure positive consumption, savings, and financial wealth along the optimal path. In that case, the state variable defined as the log of the ratio between the financial wealth and the working income is well defined and stationary[3].

Two facts are important to drive results. First, the marginal utility of consumption tends to infinity as the consumption is reduced to zero; and second, that in each period there is a strictly positive probability that the working income is zero (in case of retirement). Due to these facts, the investor chooses the optimal rules on consumption, investment, expenses with education, and savings in order to ensure a strictly positive consumption in future periods. This implies not only attaining each period with a strictly positive endowment, namely $W^{es}_{t+i} > 0, W^{ep}_{t+i} > 0$, but also that every period ends with strictly positive savings, $W^{es}_{t+i} + L_{t+i} - X_{t+i} - C^{es}_{t+i} > 0$ and $W^{ep}_{t+i} + T_{t+i} - C^{ep}_{t+i} > 0$. Therefore, the logarithms of these quantities are well defined. Similarly, in the case of retirement we must have $W^{r}_{t+i}$ and $W^{r}_{t+i} - C^{r}_{t+i} > 0$. We thus have three budget constraints to be log-linearized.

In the first case (employed investor educating himself), from equation (13) we have

$$\frac{W^{es}_{t+i+1}}{L_{t+i+1}} = \frac{L_{t+i}}{L_{t+i+1}} \left( 1 + \frac{W^{es}_{t+i}}{L_{t+i}} - \frac{X_{t+i}}{L_{t+i}} - \frac{C_{t+i}}{L_{t+i}} \right) (1 + R_{p,t+i+1}),$$

[3] Viceira (2001) demonstrates the stationarity of $\log{(W/L)}$ along the optimal path.

leading to

$$w_{t+i+1}^{es} = \log\left[1+\exp\left(w_{t+i}^{es}-l_{t+i}\right)-\exp\left(x_{t+i}-l_{t+i}\right)-\exp\left(c_{t+i}^{es}-l_{t+i}\right)\right] + l_{t+i+1}-\Delta l_{t+i+1}+r_{p,t+i+1}^{es}. \tag{19}$$

Linearizing equation (19) around the expected values of $(c_{t+i}-l_{t+i})$, $(x_{t+i}-l_{t+i})$ and $(w_{t+i}^{ep}-l_{t+i})$ leads to

$$w_{t+i+1}^{es}-l_{t+i+1} \approx k^{es}+\rho_w^{es}(w_{t+i}^{es}-l_{t+i})-\rho_x^{es}(x_{t+i}-l_{t+i}) - \rho_c^{es}(c_{t+i}^{es}-l_{t+i})-\Delta l_{t+i+1}+r_{p,t+i}^{es} \tag{20}$$

where $k^{es}$, $\rho_w^{es}$, $\rho_c^{es}$, $\rho_x^{es}$ are log-linearized constants described in Appendix A.

In the second case (employed investor promoted), we have from equation (14)

$$\frac{W_{t+i+1}^{ep}}{T_{t+i+1}}=\frac{T_{t+i}}{T_{t+i+1}}(1+\frac{W_{t+i}^{ep}}{T_{t+i}}-\frac{C_{t+i}^{ep}}{T_{t+i}})(1+R_{p,t+i+1})$$

leading to

$$w_{t+i+1}^{ep} = \log\left[1+\exp\left(w_{t+i}^{ep}-\tau_{t+i}\right)+\exp\left(c_{t+i}^{ep}-\tau_{t+i}\right)\right] + \tau_{t+i+1}-\Delta\tau_{t+i+1}+r_{p,t+i+1}^{ep}. \tag{21}$$

Linearizing equation (21) around the expected values of $(c_{t+i}-\tau_{t+i})$ and $(w_{t+i}^{ep}-\tau_{t+i})$ leads to

$$w_{t+i+1}^{ep}-\tau_{t+i+1}\approx k^{ep}+\rho_w^{ep}(w_{t+i}^{ep}-\tau_{t+i})-\rho_c^{ep}(c_{t+i}^{ep}-\tau_{t+i})-\Delta\tau_{t+i+1}+r_{p,t+i+1}^{ep} \tag{22}$$

where, $k^{ep}$, $\rho_w^{ep}$, $\rho_c^{ep}$ are log-linearized constants described in Appendix A.

Finally, for the retired investor we have from equation (15)

$$\frac{W^r_{t+i+1}}{W_{t+i}} = (1 - \frac{C_{t+i}}{W_{t+i}})(1 + R_{p,t+i+1})$$

leading to

$$w^r_{t+i+1} - w_{t+i} = \log\left[1 - \exp\{c_{t+i} - w_{t+i}\}\right] + r^r_{p,t+i+1}.$$

Linearizing this last equation around the expected value of $(c_{t+i} - w_{t+i})$ and $E(w^r_{t+i} - w_{t+i})$, we obtain

$$w^r_{t+i+1} - w_{t+i} = k^r - \rho^r_c(c_{t+i} - w_{t+i}) + r^r_{p,t+i+1} \tag{23}$$

where $k^r$, $\rho^r_c$ are log-linearized constants described in Appendix A. All these constants $k^i$, $\rho^i_w, \rho^i_c, \rho^i_x$, for $i = r, es, ep$ depend only on the long-run expected value of the logarithm of ratios between financial wealth, working income, gains of promotion, and consumption. Campbell and Viceira (1999) derive an approximate expression for the log-return of the portfolios

$$\begin{aligned}
\log(1+R) &= r \\
R_{p,t+i+1} &= \alpha R_{t+i+1} + (1-\alpha) R_f = \alpha (R_{t+i+1} - R_f) + R_f \\
r_{p,t+i+1} &= \alpha (r_{t+i+1} - r_f) + r_f + \frac{1}{2}\alpha(1-\alpha)\sigma^2_u.
\end{aligned}$$

Notice that the Euler equations (16), (17), and (18) are non-linear. However, we may find an approximate log-linear solution for each of them[4]

- In the first case, the Euler equation for optimal consumption is

$$1 = E_t\left\{\left[\beta\pi^e\pi^p\left(\frac{C^{ep}_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma} + \beta\pi^e\pi^s\left(\frac{C^{ep}_{t+i+1}}{C^{ep}_{t+i}}\right)^{-\gamma} + (1-\pi^e)\beta^r\left(\frac{C^r_{t+i+1}}{C^{ep}_{t+i}}\right)^{-\gamma}\right](1 - R_{p,t+i+1})\right\}.$$

[4]Each of these Euler equations is derived in Appendix A.

Define $A(u,v) = -\gamma E_{t+i}\left(c^{u}_{t+i+1} - c^{v}_{t+i}\right) + \frac{1}{2}var_{t+i}\left[r_{i,t++i1} - \gamma\left(c^{u}_{t+i+1} - c^{v}_{t+i}\right)\right]$ for $u = es, ep, r$ and $v = es, ep, r$. Log-linearizing the Euler equation yields

$$0 = E_{t+i}(r_{i,t+i+1}) + \pi^{e}\sum_{j=p,s}\pi^{j}\left\{\log\beta + A(ej,es)\right\} + (1-\pi^{e})\left\{\log\beta^{r} + A(r,es)\right\}. \tag{24}$$

The Euler equation for the amount spent with education is

$$\frac{1}{\beta} = E_t\{\pi^{e}\pi^{s}\frac{v'(X_{t+i+1})}{v'(X_{t+i})}(1 - R_{i,t+i+1})\},$$

becoming after log-linearization

$$0 = \pi^{e}\pi^{s}E_t\left\{\log\beta - \gamma(x_{t+i+1} - x_{t+i}) + r_{i,t+i+1} + \frac{1}{2}var_{t+i}\left[r_{i,t++i1} - \gamma(x_{t+i+1} - x_{t+i})\right]\right\}. \tag{25}$$

The Euler equation for the second case is

$$1 = E_t\left\{\left[\pi^{e}\beta\left(\frac{C^{ep}_{t+i+1}}{C^{ep}_{t+i}}\right)^{1-\gamma} + (1-\pi^{e})\beta^{r}\left(\frac{C^{r}_{t+i}}{C^{ep}_{t+i}}\right)\right](1 + R_{i,t+i+1})\right\}.$$

Log-linearizing this expression we obtain

$$0 = E_{t+i}(r_{i,t+i+1}) + \pi^{e}\left\{\log\beta + A(ep,ep)\right\} + (1-\pi^{e})\left\{\log\beta^{r} + A(r,ep)\right\}. \tag{26}$$

Finally, for the third case, the Euler equation is

$$1 = E_t\left[\beta^{r}\left(\frac{C^{r}_{t+i+1}}{C^{r}_{t+i}}\right)^{-\gamma}(1 + R_{i,t+i+1})\right]$$

and the log-linearized expression is

$$0 = \log \beta^r + E_{t+i}(r_{i,t+i+1}) + A(r, r). \tag{27}$$

With this set of approximated analytical solutions we may now characterize the optimal choices of consumption and investment.

In the next sections we shall characterize the optimal choices for investors who retired, for investors who are employed but were promoted, and finally, for those investors who did not yet receivea promotion and continue to invest in their education.

Due to the fact that retirement is irreversible, the optimal policy for retired investors does not depend on the optimal policies of employed investors. However, when the investor is employed, he or she must consider the possibility of retiring in order to decide when and how to save. A similar argument applies to the promotion process. After a promotion, the optimal policy should not depend on the optimal policy if he were still without his promotion. However, while the investor stays in the current job, he must consider the possibility of a future promotion in order to define his optimal allocation policy.

## 3.3 Choices of a retired investor

When the investor is retired the working income is zero in all future periods by assumption. Therefore, the investor makes his choices according only to his wealth. The investor faces a classical decision for which a closed form solution has existed since the papers of Merton (1969) and Samuelson (1969)). In this case, the solution method described above produces the exact solution up to the discrete-time approximation to the log return on wealth.

**Proposition 1** *The optimal rules for consumption and portfolio investment when the working in-*

*vestor is retired are:*

$$c^r_{t+i} = b^r_0 + b^r_1 w_{t+i} \tag{28}$$

$$\alpha^r = \frac{\mu + \frac{\sigma^2_u}{2}}{\gamma b^r_1 \sigma^2_u} \tag{29}$$

*where* $b^r_1 = 1$ *and*

$$b^r_0 = -\left(\frac{1}{b^r_1 \rho^r_c}\right) \cdot \left[\left(\frac{1}{\gamma} - b^r_1\right) E[r_{p,t+i+1}] + \frac{1}{\gamma}\log\beta + \frac{1}{2\gamma}(1-\gamma b^r_1)^2 var_{t+i}(r_{p,t+i+1}) - b^r_1 k^r\right].$$

**Proof.** See Appendix B ■

This Proposition presents the optimal portfolio rule as depending on the parameters of the consumption function. We must then start characterizing the optimal consumption in this case, and then move to the analysis of the optimal portfolio.

### 3.3.1 Optimal consumption

The above Proposition 1 shows that the logarithm of the consumption is a linear function of the state variable given by the logarithm of the financial wealth. The slope of this relationship $(b^r_1)$ is the elasticity of the consumption with respect to the financial wealth. This elasticity is exactly one in the retirement state, implying a constant ratio consumption/wealth during the retirement period.

The intercept $b^r_0$ characterizes the constant consumption-wealth ratio of retired investors. This term shows that consumption is affected by several factors. The relative consumption increases with the discount rate. It also increases with the portfolio's expected return for those retired investors whose elasticity of intertemporal rate of substitution is less than the elasticity of consumption with respect to his or her wealth $(\frac{1}{\gamma} - b^r_1) < 0$. On the other hand, if the retired investor has an intertemporal elasticity greater than the consumption elasticity $(\frac{1}{\gamma} - b^r_1) > 0$, the consumption will tend to decrease since the investors choose to save more, given that the substitution effect dominates. There also exists a precautionary saving effect given by the variance of the equation above. The magnitude of this effect is proportional to the coefficient of relative risk aversion and wealth elasticity of consumption.

### 3.3.2 Portfolio choice

Proposition 1 also characterizes $\alpha^r$, the optimal proportion of wealth to be invested by the retired investor in the risky asset. Notice that there is only one component in the expression for $\alpha^r$. It represents the allocation that would be optimal if there were no income being received by the investor. This term is proportional to the asset risk premium and decreasing with the coefficient of risk aversion $\gamma$. Also, it decreases with the consumption elasticity $b_1^r$. This term is known as the *myopic portfolio rule* (see Campbell and Viceira, among others).

## 3.4 Choices of a promoted investor

In the case studied in this section, the investor faces the uncertainty about being still employed in the next period. Since the investor was already promoted, he makes his choice based on the working income, the extra amount of wage that he received with the promotion, and consumption. We may now characterize the optimal allocation under such circumstances.

**Proposition 2** *The optimal rules for consumption and optimal portfolio investment when the working investor is promoted are:*

$$c_{t+i}^{ep} - \tau_{t+i} = b_0^{ep} + b_1^{ep}(w_{t+i} - \tau_{t+i}) \tag{30}$$

$$\alpha^{ep} = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^c \sigma_u^2} - \frac{\pi^e(1 - b_1^{ep})\sigma_{\xi u}}{b_1^c \sigma_u^2} - \frac{\pi^e(1 - b_1^{ep})(z-1)\sigma_{\xi u}}{b_1^c \sigma_u^2} \tag{31}$$

*where* $b_1^c = \pi^e b_1^{ep} + (1-\pi^e) b_1^r$ *and* $0 < b_1^{ep} < 1$ *with*

$$\begin{aligned} b_0^{ep} &= -\frac{1}{[(1-\pi^e)+\rho_c^{ep} b_1^c]}[\left(\frac{1}{\gamma} - b_1^c\right) E_{t+i}(r_{p,t+i+1}) + \frac{1}{\gamma}(\pi^e \log \beta + (1-\pi^e)\log \beta^r) \\ &\quad + \frac{1}{2\gamma} V^{ep} + (b_1^c - 1) zg - b_1^c k^{ep} - (1-\pi^e) b_0^r]. \end{aligned}$$

**Proof.** See Appendix B[5]. ■

Just as in Proposition 1, the above result shows that the optimal rule for the portfolio allocation depends on parameters of the optimal consumption rule.

### 3.4.1 Optimal Consumption

Proposition 2 shows that the logarithm of the ratio consumption/new income is a linear function of the logarithm of the state variable, the ratio wealth/new working income. The slope $b_1^{ep}$ of this relationship is the elasticity of the consumption with respect to the financial wealth in the state where the investor is employed and promoted. Similarly, $1 - b_1^{ep}$ is the elasticity of consumption with respect to the new working income. Moreover, since we know that the gains of promotion are proportional to the working income $(\tau = zl)$, then the elasticity of the consumption with respect to the promotion gain is $(1 - b_1^{ep})(z-1)$.

The exponential of the intercept of the linear relationship $(b_0^{ep})$ is a factor scaling the optimal ratio consumption/working income above or below the current ratio between wealth and working income. As in the case of the retired investor, the optimal consumption is affected by several factors. First, relative consumption is increasing in the interest rate. The relative consumption increases with the discount rate. It also increases with the expected return of the optimal portfolio for retired investors who have intertemporal substitution elasticity lower than the elasticity of consumption

[5] In Viceira (2001), the term $b_1^c k^{ep}$ in our last line reads simply $k^{ep}$. This is responsible for the main differences between our simulation when we make $z = 1$ to recover his case, and his simulation. Qualitatively, however, the results do not change.

with respect to wealth $\left(\frac{1}{\gamma} - b_1^{el}\right) < 0$. If the reverse happens, however, and $\left(\frac{1}{\gamma} - b_1^{el}\right) > 0$, consumption will tend to diminish, since investors will prefer to save more, given that the substitution effect dominates. There also exists a precautionary saving effect given by the variance of the above equation. Finally, $b_0^{ep}$ is an increasing function of the variability of the working income $(g)$ and on the gains of promotion $(z)$. An investor who expects his or her working income to increase in the future, may feel freer to consume a larger fraction of his or her current resources.

### 3.4.2 Portfolio Choice

Proposition 2 also characterizes the optimal proportion $\alpha^{ep}$ of wealth invested in risky assets. Notice that there are three main components in this equation. The first represents how the allocation would be if the working income had no impact and if there were no promotion at all. This first term is proportional to the risk premium of the risky asset and proportional to the inverse to the risk aversion coefficient $\gamma$, and to the elasticity of consumption with respect to the wealth. It is important to notice that the consumption/wealth elasticity in this equation is given by $b_1^c = \pi^e b_1^{ep} + (1 - \pi^e) b_1^r$, an average elasticity considering the different states of nature. In particular, $b_1^{ep} < b_1^r$ implies that employed investors are less risk averse than retired investors (once that $b_1^r = 1$ and $0 < b_1^{ep} < 1$).

The second term explains the relationship between the working income and allocation in risky assets. If this term is different from zero it is because somehow the working income is related to the return on the risky asset. If there is a positive covariance, the working income is increasing the risk of the investor. In that case, the investor will seek to diminish the risk, lowering his or her investments in the risky asset. In the opposite case of a negative covariance, working income can be seen as an insurance against the position on the risky asset. In that case, investors may increase their position in the risky asset. That is why this second term is known as the *hedging term*.

The third term describes the position of the investor reflecting how the return of the risky asset is correlated with the gains of education provided by the promotion. Notice that the gain is proportional to the investor working income. Therefore, this term is non-zero as long as the working income covaries with the return of the risky asset. Notice also that the coefficient $z$ denoting the fraction of income that is offered to the agent as a rise of his wage is proportional to the inverse to the fraction of wealth invested in the risky asset when the considered covariance is positive. In other words, in that case the rise would be increasing the risk and the investor tends to compensate that by investing less in the risky asset. Clearly the gains of promotion will act as an insurance against the risky asset in the opposite case of a negative covariance, and the larger the fraction of gains, the larger will be the proportion of wealth invested in the risky asset.

**Proposition 3** *When the working income is independent of the return on the risky asset, promoted investors invest a higher fraction of their savings in the risky asset than do retired investors. Moreover,* $\lim_{\pi^e \to 0} \alpha^{ep} = \alpha^r$.

**Proof.** Take $\sigma_{\xi u} = 0, \alpha^{ep} = \frac{\left(\mu + \frac{\sigma_u^2}{2}\right)}{\left(\gamma b_1^c \sigma_u^2\right)}$ and $\alpha^r = \frac{\left(\mu + \frac{\sigma_u^2}{2}\right)}{\left(\gamma b_1^r \sigma_u^2\right)}$. Since $b_1^c = \pi^e b_1^{ep} + (1 - \pi^e) b_1^r$ and $b_1^{ep} < b_1^r$, it follows that $b_1^c < b_1^r \Rightarrow \alpha^{ep} > \alpha^r$. Besides, when $\pi^e \to 0 \Rightarrow b_1^c \to b_1^r$, leading to $\alpha^{ep} \to \alpha^r$. ■

Intuitively, when the investor is retired he will have one less source of income than in the case where he is working. Therefore, the effect in consumption of an increase in wealth would be relatively greater to the retired investor, and his consumption/wealth elasticity would be larger. Thus, considering the idiosyncratic risk of working income, the allocation in risky assets is relatively greater for the employed than for the retired investor. This Proposition shows that with non-stochastic working income, human capital is equivalent to an implicit investment in a riskless asset, allowing to increase the proportion of wealth invested in the risky asset.

## 3.5 Choices of an employed investor spending in education

As in the previous section, since the investor is employed he faces the possibility of two different states in the future. Either the investor is employed in the next period with probability $\pi^e$, receiving a working income, or the investor is retired with probability $1-\pi^e$, receiving no working income. However, there is an additional source of uncertainty, namely whether he will be promoted in the next period or not. For simplicity, we assume that the chance of promotion is greater than the possibility of retirement of the agent, that is:

$$\frac{1}{\pi^r} > \frac{1}{\pi^p} \Rightarrow \pi^p > (1-\pi^e).$$

The uncertainty about whether or not the agent will be promoted in the next period exists only while the investor is employed. Given all these considerations, the employed investor will then decide on the consumption level, the optimal portfolio of financial assets, and how much to spend with his education looking for future promotion (or better job).

We may now characterize the optimal rules characterizing those decisions.

**Proposition 4** *The optimal rules for the consumption of the investor, for how much is spent with education and for the investment in the portfolio when the investor is working are*

$$c^{es}_{t+i} - l_{t+i} = b^{es}_0 + b^{es}_1 (w_{t+i} - l_{t+i}) \tag{32}$$

$$x_{t+i} - l_{t+i} = b^{es}_2 + b^{es}_3 (w_{t+i} - l_{t+i}) \tag{33}$$

$$\alpha^{es} = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b^{cc}_1 \sigma^2_u} - \frac{\pi^e \pi^s (1-b^{es}_1)\sigma_{\xi u}}{b^{cc}_1 \sigma^2_u} - \frac{\pi^e \pi^p (1-b^{ep}_1) z \sigma_{\xi u}}{b^{cc}_1 \sigma^2_u} \tag{34}$$

*where* $0 < b^{es}_1, b^{es}_3 < 1, b^{cc}_1 = \pi^e(\pi^p b^{ep}_1 + \pi^s b^{es}_1) + (1-\pi^e) b^r_1$ *and*

$$\begin{aligned} b^{es}_0 \quad = \quad & -\frac{1}{[(1-\pi^e\pi^s) + b^{cc}_1 \rho^{es}_c]} [\left(\frac{1}{\gamma} - b^{cc}_1\right) E_{t+i}(r_{p,t+i+1}) + \frac{1}{\gamma}(\pi^e \log\beta + (1-\pi^e)\log\beta^r) + \\ & \frac{1}{2\gamma} V^{es} + b^{cc}_1 (k^e - b^{es}_2 \rho^{es}_x) - g(b^{cc}_1 - 1) - (\pi^e \pi^p b^{ep}_0 + (1-\pi^e b^r_0)) - \pi^e \pi^p (1-b^{ep}_1)(z-1) g] \end{aligned}$$

$$
\begin{aligned}
b_2^{es} &= -\frac{1}{(1+b_1^{es}\rho_x^{es})}[\left(\frac{1}{\gamma}-b_1^{es}\right)E_{t+i}(r_{p,t+i+1})+\log\beta-g(1-b_1^{es})+ \\
&+\frac{1}{2}var_{t+i}(r_{p,t++i1}-\gamma(c_{t+i+1}^{es}-x_{t+i}))-b_0^{es}+b_1^{es}(k^{es}-\rho_c^{es}b_0^{es})].
\end{aligned}
$$

**Proof.** See Appendix B. ■

This Proposition shows that the optimal portfolio rule in the case under study depends on the parameters of the functions describing the consumption and spending with education. We shall analyze each of these functions.

### 3.5.1 Optimal Consumption

Proposition 4 shows that the logarithm of the ratio consumption/working income is a linear function of the logarithm of the state variable, the ratio wealth/working income. The slope of this relationship $(b_1^{es})$ is the elasticity of the consumption with respect to the financial wealth in the state where the investor is employed with no promotion. Similarly, $1-b_1^{es}$ is the elasticity of consumption with respect to the working income.

The exponential of the intercept of the linear relationship $(b_0^{es})$ is a factor scaling the optimal ratio consumption/working income above or below the current ratio between wealth and working income. As in both cases studied above, the optimal consumption is affected by several factors. First, relative consumption is decreasing in $\beta$. It also increases with the expected return of the optimal portfolio for investors who have intertemporal substitution elasticity lower than the elasticity of consumption with respect to wealth $\left(\frac{1}{\gamma}-b_1^{es}\right)<0$. If the reverse happens, however, and $\left(\frac{1}{\gamma}-b_1^{es}\right)>0$, consumption will tend to diminish, since investors will prefer to save more, given that the substitution effect dominates. There also exists a precautionary saving effect given by the variance of the above equation. Finally, $b_0^{es}$ is an increasing function of the variability of the working income $g$. An investor who expects his working income to increase in the future, may feel freer to consume a larger fraction of his current resources.

In addition, notice that $b_2^{es}$ is the intercept of the equation characterizing the amount spent with education. Hence, the more is spent with the education, the less the investor consumes.

### 3.5.2 Optimal amount spent with education

Proposition 4 shows that the logarithm of the ratio between the amount spent with education and the working income is a linear function of the logarithm of the state variable, the ratio wealth/working income. The slope of this relationship $(b_3^{es})$ is the elasticity of the amount spent with education with respect to the financial wealth in the state where the investor is employed without promotion. Similarly, $1 - b_3^{es}$ is the elasticity of amount spent with education with respect to the working income.

The exponential of the intercept of the linear relationship $(b_2^{es})$ is a factor scaling the optimal ratio between the amount spent with education and the working income above or below the current ratio between wealth and working income. As above, the optimal consumption is affected by several factors. First, the relative amount spent decreases with $\beta$. It also increases with the expected return of the optimal portfolio for investors who have intertemporal substitution elasticity lower than the elasticity of consumption with respect to wealth $\left(\frac{1}{\gamma} - b_1^{es}\right) < 0$. If the reverse happens, however, and $\left(\frac{1}{\gamma} - b_1^{es}\right) > 0$, consumption will tend to diminish, since investors will prefer to save more, given that the substitution effect dominates. There also exists a precautionary saving effect given by the variance of the above equation. Finally, $b_2^{el}$ is an increasing function of the variability of the working income. An investor who expects his working income to increase in the future, may feel freer to consume a larger fraction of his or her current resources.

### 3.5.3 Optimal portfolio

Proposition 4 also characterizes the optimal proportion $\alpha^{es}$ of wealth invested in risky assets in this specific case. Notice that there are three main components in this equation. The first component represents how the allocation would be if the working income had no impact. This first term is proportional to the risk premium of the risky asset and proportional to the inverse to the risk aversion coefficient $\gamma$, and to the elasticity of consumption with respect to the wealth. It is important to notice that the consumption/wealth elasticity in this equation is $b_1^{cc} = \pi^e(\pi^p b_1^{ep} + \pi^s b_1^{es}) + (1 - \pi^e)b_1^r$, an average elasticity considering the different states of nature. In particular, since $b_1^e = \pi^p b_1^{ep} + \pi^s b_1^{es} < b_1^r$, this would imply that employed investors are less risk averse than retired investors.

The second term explains the relationship between the working income and allocation in risky assets. Just as before, if this term is different from zero it is because somehow the working income is related to the return on the risky asset. If there is a positive covariance, the working income is increasing the risk of the investor. In that case, the investor will tend to diminish the risk, lowering his or her investments in the risky asset. In the opposite case of a negative covariance, working income can be seen as an insurance against the position on the risky asset. In that case, investors may increase their position in the risky asset. Again, this second term is known as the *hedging term*.

The third term describes the position of the investor reflecting how the return of the risky asset is correlated with his gains of education that will be provided by the promotion in the future. Notice that, although we are considering the case where the promotion has not yet occurred, the investor must make his choices strategically, knowing that at some point in the future his salary may rise. Also, recall that the gain is proportional to the investor working income. Therefore, this term is non-zero as long as the working income covaries with the return of the risky asset. Notice also that the coefficient $z$ denoting the fraction of income that is added to his wage is proportional to the inverse to the fraction of wealth invested in the risky asset when the considered covariance is positive. In other words, in that case the gain would be increasing the risk and the investor tends to compensate that by investing less in the risky asset. Again, it is clear that it will act as an insurance against the risky asset in the opposite case of a negative covariance, and the larger the fraction of gain, the larger will be the proportion of wealth invested in the risky asset.

**Proposition 5** *When the working income is independent of the return on the risky asset, working investors who are still spending in their education invest a lower fraction of their savings in the risky asset than promoted investors if* $b_1^{es} > b_1^{ep}$*, and invest a greater fraction if* $b_1^{es} < b_1^{ep}$*. Additionally, working investors invest a larger fraction of their savings in risky assets than do retired investors. Moreover,* $\lim_{\pi^s \to 0} \alpha^{es} = \alpha^{ep}$*, and* $\lim_{\pi^e \to 0} \alpha^{es} = \alpha^r$.

**Proof.** Take $\sigma_{\xi u} = 0, \alpha^{es} = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^{cc} \sigma_u^2}, \alpha^{ep} = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^{c} \sigma_u^2}$, and $\alpha^r = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^{r} \sigma_u^2}$. Since $b_1^{cc} = \pi^e(\pi^p b_1^{ep} + \pi^s b_1^{es}) + (1 - \pi^e) b_1^r$ and $b_1^c = \pi^e b_1^{ep} + (1 - \pi^e) b_1^r$ if $b_1^{es} < b_1^{ep} \Rightarrow b_1^{cc} < b_1^c \Rightarrow \alpha^{es} > \alpha^{ep}$. On the other hand, if $b_1^{es} > b_1^{ep} \Rightarrow b_1^{cc} > b_1^c \Rightarrow \alpha^{es} < \alpha^{ep}$. Similarly, since $b_1^{cc} = \pi^e(\pi^p b_1^{ep} + \pi^s b_1^{es}) + (1 - \pi^e) b_1^r$, once $b_1^e < b_1^r \Rightarrow b_1^{cc} < b_1^r \Rightarrow \alpha^{es} > \alpha^r$. Besides, when $\pi^s \to 0 \Rightarrow b_1^{cc} \to b_1^c$ and $\alpha^{es} \to \alpha^{el}$. Moreover, $\pi^e \to 0 \Rightarrow b_1^{cc} \to b_1^r$ and $\alpha^{es} \to \alpha^r$. ■

Different from Proposition 3, it is not clear if $b_1^{ep} < (>) b_1^{es} (b_1^c < (>) b_1^{cc})$. However, since the working income path is longer to the investors at this stage of life, and they expect their income to rise, it is natural that their consumption/wealth elasticity be lower than in forward stages of life. Then, we can expect that in the early phase of life, when the investor is educating himself, he makes more investment in the risky asset.

Therefore, just as before, this Proposition shows that with non-stochastic working income, human capital is equivalent to an implicit investment in a riskless asset, allowing to increase the proportion of wealth invested in the risky asset.

# 4 Calibration

In this section we illustrate the analytical results presented above. A calibration exercise is based on an empirically plausible parameterization of the process for asset returns and the process for individual labor income[6].

The optimal policies in the periods pre- and post-promotion depend on $\rho_x^{es}$, $\rho_c^{es}$, $\rho_w^{es}$, $\rho_c^{ep}$, and $\rho_w^{ep}$ respectively, and the log-linerization parameters constants in the budget constraint (equations (20) and (22)). But these constants are endogenous parameters, because they are monotonic functions of the mean financial ratio wealth-labor income. In fact, equations (30), (31), and (32), (33), (34) for the optimal policies and the equations for the constants (given in the Appendix) define a non-linear mapping of the constants onto themselves.

We solve for the fixed point in this mapping using a recursive algorithm. First we define a set of parameter values, next we compute $b_0^r$ and $\alpha^r$ (which do not depend on any log-linearization constants), and we choose initial values for the log-linearization constants[7]. Then, we compute $b_0^{ep}, b_1^{ep}, b_0^{es}, b_1^{es}, b_2^{es}, b_3^{es}, \alpha^{ep}$, and $\alpha^{es}$, using them to obtain a new set of values fo the log-linerization parameters constants in the budget constraint. From this new set of constants, we can find new values for the optimal policies in the employment state. This recursion continues until the convergence is achieved.

The values for the parameters describing the investment opportunity set are based on the historical estimates of the average equity premium, the short-term real interest rate, and the variance of excess stock returns in the U.S. stock market[8]. The return of the riskless asset $R_f$ is set to 2% per year. The standard deviation of unexpected log excess returns $(\sigma_u)$ is set to 18% per year. The log excess return on the risky asset $(\mu)$ is set to 4.21% per year to match the historical 6% excess return on equities. The values for the parameters describing the labor income process are based on the most recent microeconometric estimates of equation $L_{t+1} = L_t \exp\{g + \xi_{t+1}\}$. The standard deviation of innovations in log labor income $(\sigma_\xi)$ is set to 10% per year. Expected log

[6]The algorithm and the parameters are the same used by Viceira (2001).

[7]We choose the initial values for the constants so that $\rho_x^{es}, \rho_c^{es}, \rho_w^{es} > 0, \rho_c^{ep}, \rho_w^{ep} > 0$ and $1 - \rho_w^{es} + \rho_c^{es} + \rho_x^{es} > 0, 1 - \rho_w^{ep} + \rho_c^{ep} > 0$. This ensures that $k^{es}$ and $k^{ep}$ are defined.

[8]Campbell, Lo, and MacKinlay (1997).

income growth $(g)$ is set so that the expected income growth equals 3% per year. After promotion, the agent has an wage upgrade of 50%, which means that $h = 0.405$.

Two different values are considered for correlation between innovations in log labor income and innovations in stock returns: 0% and 25%. The zero correlation value represents the important benchmark case of idiosyncratic labor income risk, and 25% correlation value is useful to illustrate the interaction of hedging and risk aversion on optimal portfolio demand.

Tables I and II report the optimal policies for relative risk aversion coefficients $\gamma = 3$ and $5$, expected number of years until retirement $\{35, 30, 25, 20, 15, 10, 5\}$, and expected number of years until promotion $\{15, 10, 5\}$. Table I presents the results when labor income is idiosyncratic, and Table II presents results when there is a 25% correlation between unexpected stock returns and shocks to labor income. The expected lifetime after retirement is set to 10 years, and the time preference rate is set to 10% per year.

Table I - Proportion of wealth invested in risky portfolio (Correlation Zero)

| Expected time until promotion | | 15 | 10 | 5 | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Expected time until retirement | | 35 | 30 | 25 | 20 | 15 | 10 | 5 | Retired |
| RRA | | | | | | | | | |
| 3 | with educ. | 82.43% | 80.18% | 79.42% | 72.42% | 69.61% | 66.47% | 62.33% | 59.98% |
| | without educ. | 81.17% | 78.15% | 75.19% | 72.42% | 69.61% | 66.47% | 62.33% | 59.98% |
| 5 | with educ. | 40.52% | 40.79% | 42.10% | 38.87% | 38.45% | 37.84% | 36.87% | 35.99% |
| | without educ. | 39.59% | 39.39% | 39.17% | 38.87% | 38.45% | 37.84% | 36.87% | 35.99% |

Table I reports $\alpha^{es}$, $\alpha^{ep}$, and $\alpha^{r}$ when labor income risk is uncorrelated with stock market risk. As predicted by propositions 3 and 4, the share of stocks in savings is routinely larger in education state than in promoted state and larger in employment state than in retired state. Intuitively, when labor income risk is idiosyncratic, no retired investors choose their portfolios as if their human capital resembles a forced investment on the riskless asset, which is reinforced when the agent invest in his own education. This investment is larger for investors with longer horizons, because expected future labor income, relative to its current level, is increasing in the expected retirement horizon. Therefore, it is optimal for investors with longer horizons to hold a larger fraction of their financial wealth in stocks.

The interaction between hedging, retirement horizons, and risk aversion is illustrated in Table

II. This table shows that a small, positive correlation between labor income risk and stock market risk has significant negative effects on the optimal portfolio demand for stocks. Both tables show that the fraction invested in risky asset goes down when risk aversion increases.

Table II - Proportion of wealth invested in risky portfolio (Correlation 0.25)

| Expected time until promotion | | 15 | 10 | 5 | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Expected time until retirement | | 35 | 30 | 25 | 20 | 15 | 10 | 5 | Retired |
| RRA | | | | | | | | | |
| 3 | with educ. | 77.05% | 75.35% | 74.80% | 69.42% | 67.30% | 64.91% | 61.76% | 59.98% |
| | without educ. | 76.26% | 73.94% | 71.70% | 69.54% | 67.38% | 64.97% | 61.79% | 59.98% |
| 5 | with educ. | 38.72% | 38.90% | 39.70% | 37.70% | 37.46% | 37.05% | 36.52% | 35.99% |
| | without educ. | 38.20% | 38.08% | 37.95% | 37.76% | 37.50% | 37.07% | 36.53% | 35.99% |

Table III shows the ratio of hedging component to total demand of risky assets. In all periods, the positive correlation of human capital with stock returns makes investors reduce their exposure to risk, decreasing their investments in risky assets. Notice that for both types of risk aversion, the hedging term represents a significant fraction of total portfolio at long retirement horizons, reaching almost 7% of the total demand of risky assets.

Table III - Hedging Component

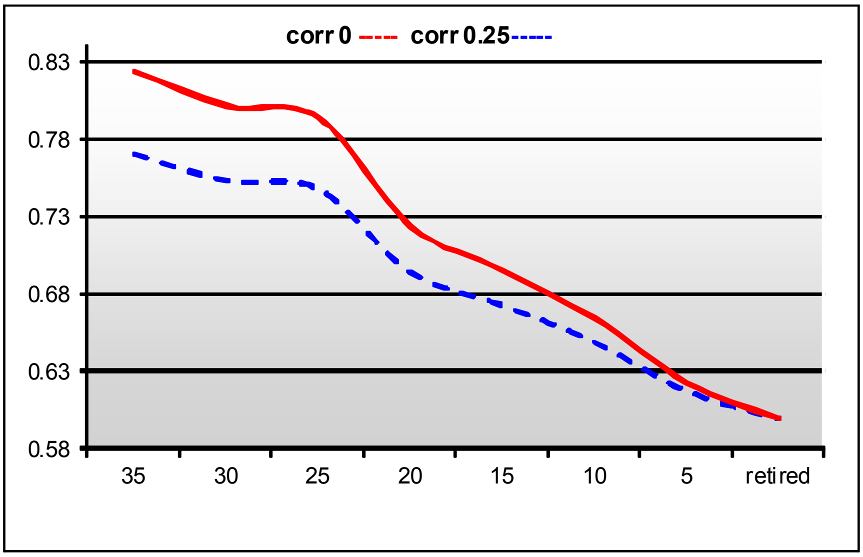
| Expected time until promotion | | 15 | 10 | 5 | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Expected time until retirement | | 35 | 30 | 25 | 20 | 15 | 10 | 5 | Retired |
| RRA | | | | | | | | | |
| 3 | with educ. | -6.98% | -6.41% | -6.18% | -4.32% | -3.43% | -2.40% | -0.92% | 0.00% |
| | without educ. | -6.44% | -6.98% | -4.87% | -4.14% | -3.31% | -2.31% | -0.87% | 0.00% |
| 5 | with educ. | -4.65% | -4.86% | -6.05% | -3.10% | -2.64% | -2.13% | -0.96% | 0.00% |
| | without educ. | -3.64% | -3.44% | -3.21% | -2.94% | -2.53% | -2.08% | -0.93% | 0.00% |

Education has a significant impact in portfolio allocation too, as predicted in preposition 4. When labor income risk is idiosyncratic, education act as a riskless investment, raising the investment in risky assets.

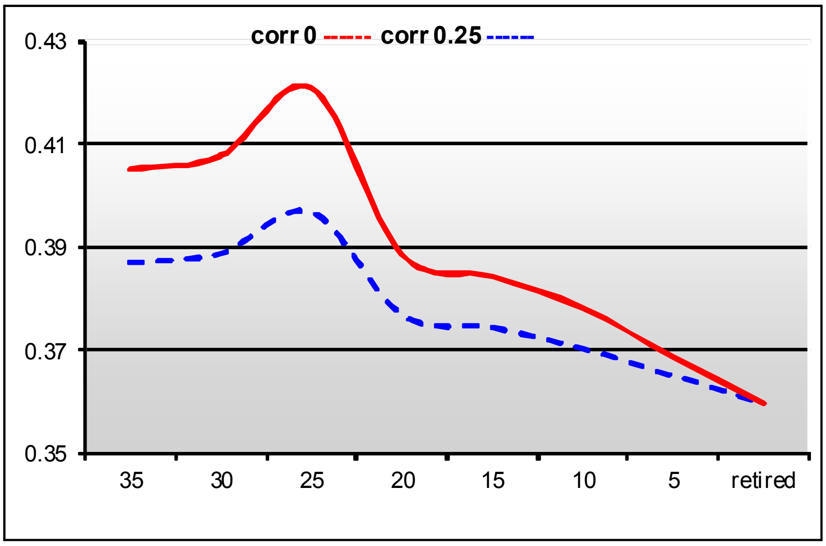
Table IV - Education Effect (correlation zero)

| | Expected time until promotion | | |
|---|---|---|---|
| | 15 | 10 | 5 |
| RRA | | | |
| 3 | 1.55% | 2.60% | 5.63% |
| 5 | 2.35% | 3.55% | 7.48% |

To illustrate the situation we report the portfolio path over the life cycle for both types of risk aversion. The red line in Figures 1 and 2 represents the proportion of demand of risky assets in portfolio when the labor income risk is idiosyncratic, and the blue line represents the same but when the correlation between labor income risk and stock return is positive. Notice that in both cases there are trends of decreasing the proportion of investment in risky assets as discussed above, and at the earlier stage, the investment in education encourages the agent to increase his position in risky assets. Also, it is easy to see the hedging effect pushing the portfolio path down.




$Figure 1 : Portfolio\ path\ over\ the\ life\ cycle\ (\gamma\ = 3)$

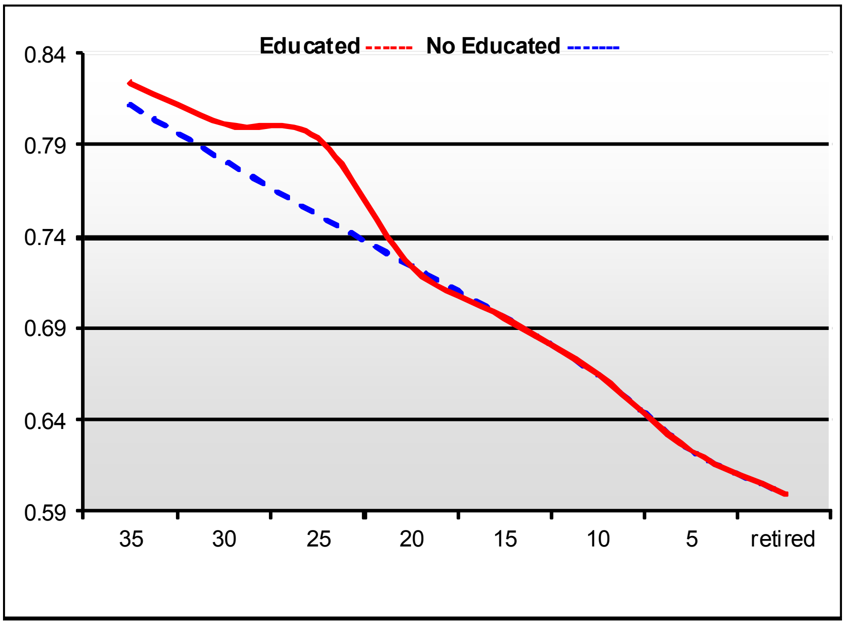



$Figure\ 2 :\ Portfolio\ path\ over\ the\ life\ cycle\ (\gamma = 5)$

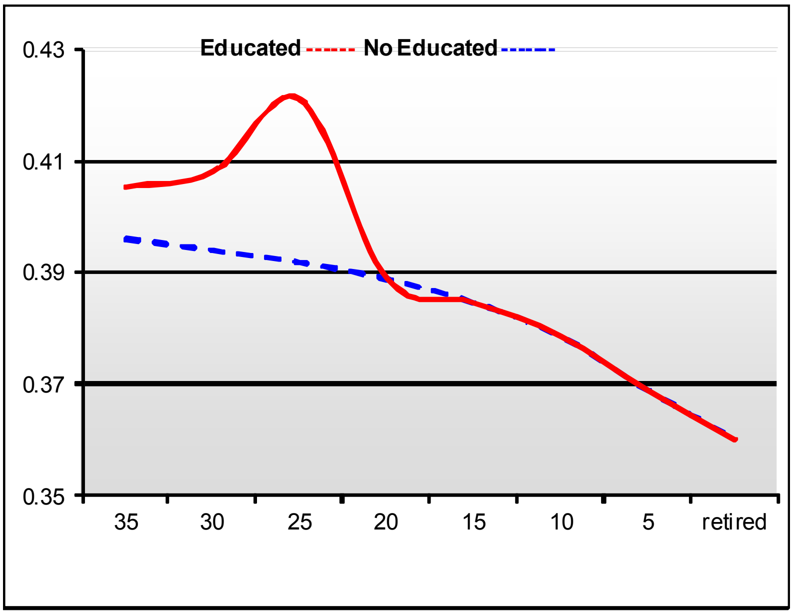
To show the education effect, the portfolio path is illustrated in both cases of risk aversion (considering zero correlation), when the investor chooses to educate himself and if not. Notice that if the choice for education occurs the investment in risky assets goes up, until the promotion occurs and the spending with education ceases. At the next stage, the allocation becomes the same as with no education[9]. If there is a positive correlation between labor income and stock returns, the

[9] In this special case with labor income being idiosyncratic.

promotion adds income and risk to the agent that educated himself, making the investor reduce his risky investment relative to the non-educated investor.




$Figure3 : Portfolio\ path\ over\ the\ life\ cycle\ (\gamma\ = 3)$




$Figure\ 4 :\ Portfolio\ path\ over\ the\ life\ cycle\ (\gamma = 5)$

# 5 Conclusions

This paper has characterized the optimal asset allocation for an investor who chooses to make an upgrade in his educational level, looking for better work opportunities in the future.

We present our main findings as follows. First, we describe how the portfolio allocation changes in time for an investor that upgrades his or her education. Next, we discuss how this allocation changes for different levels of education. In particular, when no education upgrade occurs, the results of Viceira (2001) are recovered.

We then examine the life cycle of an investor who upgraded his or her education. When the working income is idiosyncratic, *i.e.*, not interfering in the portfolio choice since the working income is not correlated with the return on the risky asset, we may state the following: since the elasticity of consumption with respect to wealth decreases as the investor retires, the fraction of wealth invested in the risky asset will be reduced. The intuition for this result is as follows. employed investors have more income sources than retired investors. Hence, the impact of a variation of employed investors' wealth on their consumption level will be lower than in the case of retired investors, and the latter will have a greater consumption/wealth elasticity than the former. Notice that investors under an educational upgrading process have an expeceted working period longer than investors already promoted. Also, such investors expect an increase in their income level in the future. It is therefore natural that their consumption/wealth elasticity be lower at this educational stage than in later stages of life (after promotion or at retirement). We then conclude that investments in the risky asset are proportionally larger during the educational period.

In this particular case where working income is idiosyncratic, we were able to obtain a decreasing curve for asset allocation in the risky asset along the different stages of the life cycle. In fact, our results point out that allocation in risky assets tend to be high in the early stages of life (investors spending money with education until their period of promotion), decreasing in the late stages of life (employed and promoted investors, until they retire).

Additionally, the life cycle characterization should be completed with the case where shocks in the working income are correlated with unexpected returns of the risky asset. In this case, an investor could change his or her risky asset allocation in order to profit from hedging opportunities, regardless of the stage of life except, of course, for retired investors who do not receive any more working income. If such correlation is positive, investors will lower the demand for risky assets as compared to the idiosyncratic case, since the working income is already risky. As correlation increases, it is optimal for employed investors to decrease the fraction of their wealth allocated in the risky asset. A sufficiently large correlation could lead the hedging term to dominate the optimal allocation, leading the investor to avoid the risky asset. Notice that, due to positive correlations, the curve for asset allocation in the risky asset along the different stages of the life cycle may present an increasing shape, as opposed to the case where working income is idiosyncratic. We are left to consider the case of negative corelation. It follows from the above considerations that negative correlations can, at most, reinforce the idiosyncratic decreasing shape of the curve for asset allocation in the risky asset along the life cycle.

We now turn to the analysis of the optimal allocation on the risky asset during the working period, comparing the case of educated and non-educated investors. Two factors affect the optimal asset allocation for an educated investor: first, the expected future gain from promotion; second, additional hedging capacity, due to a higher income level. The increase in expected income level makes the consumption/wealth elasticity decrease, contributing to an increase in the proportion of wealth invested in the risky asset. The additional hedging capacity depends directly on the covariance between the working income and the return of the risky asset. If this covariance is positive, education may be seen as a risky investment, reducing the proportion of wealth invested in the risky asset. This last effect may even dominate the effect of the first factor when the covariance is too high. If the covariance is negative, the investment in education can be seen as an insurance, leading investors being educated to increase their share of risky assets, as compared to non-educated investors.

# A Log-linearization of the budget restriction and of the Euler equations

## A.1 Log-linearization of the budget restriction of the retired agent

The budget restriction of a retired agent corresponds to the equation (23), and may be written as

$$
\begin{aligned}
\frac{W^r_{t+i+1}}{W_{t+i}} &= (1 - \frac{C_{t+i}}{W_{t+i}})(1 + R_{p,t+i+1}) \\
&\Rightarrow\ w^r_{t+i+1} - w_{t+i} = \log\left[1 - \exp\{c_{t+i} - w_{t+i}\}\right] + r^r_{p,t+i+1}.
\end{aligned}
$$

Linearizing the logarithm of this equation, taking the first-order Taylor expansion around $(c^r_{t+i} - w_{t+i}) = E(c^r_{t+i} - w_{t+i})$ and $(w^r_{t+i} - w_{t+i}) = E(w^r_{t+i} - w_{t+i})$, we have

$$
w^r_{t+i+1} - w_{t+i+1} \approx k^r - \rho^r_c(c_{t+i} - w_{t+i}) + r^r_{p,t+i+1}
$$

where

$$
\begin{aligned}
\rho^r_c &= \frac{\exp E\left(c^r_{t+i} - w_{t+i}\right)}{1 - \exp E\left(c^r_{t+i} - w_{t+i}\right)} \\
k^r &= -(1 + \rho^r_c)\log(1 + \rho^r_c) + \rho^r_c \log(\rho^r_c)
\end{aligned}
$$

where $\rho^r_c > 0$, since $W_t - C_t > 0$ through the optimal path.

## A.2 Log-linearization of the budget restriction of a promoted working agent

When the working agent has no more expenditures with education, the budget constraint is given by equation (14), which may be rewritten as

$$\begin{aligned}
\frac{W_{t+i+1}^{ep}}{T_{t+i+1}} &= \frac{T_{t+i}}{T_{t+i+1}}(1+\frac{W_{t+i}^{ep}}{T_{t+i}}-\frac{C_{t+i}^{ep}}{T_{t+i}})(1+R_{p,t+i+1}) \\
&\Rightarrow \; w_{t+i+1}^{ep}-\tau_{t+i+1} = \log\left[1+\exp\{w_{t+i}^{ep}-\tau_{t+i}\}-\exp\{c_{t+i}^{ep}-\tau_{t+i}\}\right] \\
&\quad -\Delta\tau_{t+i+1}+r_{p,t+i+1}^{ep}.
\end{aligned}$$

Linearizing the logarithm of this equation, taking the first-order Taylor expansion around $(c_{t+i}^{ep}-\tau_{t+i}) = E(c_{t+i}^{ep}-\tau_{t+i})$ and $(w_{t+i}^{ep}-\tau_{t+i}) = E(w_{t+i}^{ep}-\tau_{t+i})$ we have

$$w_{t+i+1}^{ep}-\tau_{t+i+1} \approx k^{ep}+\rho_w^{ep}(w_{t+i}^{ep}-\tau_{t+i})-\rho_c^{ep}(c_{t+i}-\tau_{t+i})-\Delta\tau_{t+i+1}+r_{p,t+i+1}^{ep}$$

where

$$\begin{aligned}
\rho_w^{ep} &= \frac{\exp E\left(w_{t+i}^{ep}-\tau_{t+i}\right)}{1+\exp E\left(w_{t+i}^{ep}-\tau_{t+i}\right)-\exp E\left(c_{t+i}^{ep}-\tau_{t+i}\right)}, \\
\rho_c^{ep} &= \frac{\exp E\left(c_{t+i}^{ep}-\tau_{t+i}\right)}{1+\exp E\left(w_{t+i}^{ep}-\tau_{t+i}\right)-\exp E\left(c_{t+i}^{ep}-\tau_{t+i}\right)}, \\
k^{ep} &= -(1-\rho_w^{es}+\rho_c^{es})\log(1-\rho_w^{es}+\rho_c^{es})-\rho_w^{es}\log(\rho_w^{es})+\rho_c^{es}\log(\rho_c^{es}).
\end{aligned}$$

## A.3 Log-linearization of the budget restriction of the working agent with expenditure with education

The budget restriction of the working agent while there are expenditures with education is given by equation (13), which may be rewritten as

$$
\begin{aligned}
\frac{W^{es}_{t+i+1}}{L_{t+i+1}} &= \frac{L_{t+i}}{L_{t+i+1}}(1+\frac{W^{es}_{t+i}}{L_{t+i}}-\frac{X_{t+i}}{L_{t+i}}-\frac{C_{t+i}}{L_{t+i}})(1+R_{p,t+i+1}) \\
&\Rightarrow\ w^{es}_{t+i+1}-l_{t+i+1} = \log\left[1+\exp\{w^{es}_{t+i}-l_{t+i}\}-\exp\{x_{t+i}-l_{t+i}\}-\exp\{c^{es}_{t+i}-l_{t+i}\}\right] \\
&\quad -\Delta l_{t+i+1}+r^{es}_{p,t+i+1}.
\end{aligned}
$$

Linearizing the logarithm of this equation, taking the first-order Taylor expansion around $(c^{es}_{t+i}-l_{t+I})=E(c^{es}_{t+i}-l_{t+i})$, $(w^{es}_{t+i}-l_{t+i})=E(w^{es}_{t+i}-l_{t+i})$ and $(x_{t+i}-l_{t+i})=E(x_{t+i}-l_{t+i})$ we have

$$
w^{es}_{t+i+1}-l_{t+i+1} \approx k^{es}+\rho^{es}_w(w^{es}_{t+i}-l_{t+i})-\rho^{es}_x(x_{t+i}-l_{t+i})-\rho^{es}_c(c^{es}_{t+i}-l_{t+i})-\Delta l_{t+i+1}+r^{es}_{p,t+i}
$$

where

$$
\begin{aligned}
\rho^{es}_w &= \frac{\exp E\left(w^{es}_{t+i}-l_{t+i}\right)}{1+\exp E\left(w^{es}_{t+i}-l_{t+i}\right)-\exp E(x_{t+i}-l_{t+i})-\exp E\left(c^{es}_{t+i}-l_{t+i}\right)}, \\
\rho^{es}_x &= \frac{\exp E\left(x_{t+i}-l_{t+i}\right)}{1+\exp E\left(w^{es}_{t+i}-l_{t+i}\right)-\exp E(x_{t+i}-l_{t+i})-\exp E\left(c^{es}_{t+i}-l_{t+i}\right)}, \\
\rho^{es}_c &= \frac{\exp E\left(c^{es}_{t+i}-l_{t+i}\right)}{1+\exp E\left(w^{es}_{t+i}-l_{t+i}\right)-\exp E(x_{t+i}-l_{t+i})-\exp E\left(c^{es}_{t+i}-l_{t+i}\right)}, \\
k^{es} &= -(1-\rho^{es}_w+\rho^{es}_x+\rho^{es}_c)\log(1-\rho^{es}_w+\rho^{es}_x+\rho^{es}_c)-\rho^{es}_w\log(\rho^{es}_w)+ \\
&\quad +\rho^{es}_x\log(\rho^{es}_x)+\rho^{es}_c\log(\rho^{es}_c).
\end{aligned}
$$

## A.4 Log-linearization of the Euler Equation for a working agent

- The Euler equation for the case where the agent is working and spending with education is

$$1 = E_t\left[\beta\left(\pi^e\left(\pi^p\left(\frac{C^{ep}_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma}+\pi^s\left(\frac{C^{es}_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma}\right)+(1-\pi^e)\beta^r\left(\frac{C^{r}_{t+i+1}}{C^{es}_{t+i}}\right)^{-\gamma}\right)(1-R_{p,t+i+1})\right].$$

Taking the logarithm and exponential in the equation between square brackets we have

$$\begin{aligned}1 &= \pi^e\pi^p E_{t+i}[\exp(\log\beta-\gamma(c^{ep}_{t+i+1}-c^{es}_{t+i})+r_{i,t+i+1})]+\\ &+\pi^e\pi^s E_{t+i}[\exp(\log\beta-\gamma(c^{es}_{t+i+1}-c^{es}_{t+i})+r_{i,t+i+1})]+\\ &+(1-\pi^e)E_{t+i}[\exp(\log\beta^r-\gamma(c^{r}_{t+i+1}-c^{es}_{t+i})+r_{i,t+i+1})]\end{aligned}$$

$$1 = \pi^e\pi^p E_{t+i}[\exp(X_{t+i+1})]+\pi^e\pi^s E_{t+i}[\exp(Y_{t+i+1})]+(1-\pi^e)E_{t+i}[\exp(Z_{t+i+1})].$$

Making a second-order expansion of $\exp(X_{t+i+1}), \exp(Y_{t+i+1})$ and $\exp(Z_{t+i+1})$ around their respective expected values $\overline{X} = E(\exp(X_{t+i+1})), \overline{Y} = E(\exp(Y_{t+i+1}))$ and $\overline{Z} = E(\exp(Z_{t+i+1}))$ we may write

$$\begin{aligned}1 &\approx \pi^e\pi^p E_{t+i}[\exp(\overline{X})(1+(X_{t+i+1}-\overline{X})+\frac{1}{2}(X_{t+i+1}-\overline{X})^2)]+\\ &+\pi^e\pi^s E_{t+i}[\exp(\overline{Y})(1+(Y_{t+i+1}-\overline{Y})+(Y_{t+i+1}-\overline{Y})^2)]\\ &+(1-\pi^e)E_{t+i}[\exp(\overline{Z})(1+(Z_{t+i+1}-\overline{Z})+(Z_{t+i+1}-\overline{Z})^2)]\end{aligned}$$

$$
\begin{aligned}
1 &\approx \pi^e \pi^p \exp(\overline{X})(1 + \frac{1}{2} var_{t+i}(X_{t+i+1})) + \\
&+ \pi^e \pi^s \exp(\overline{Y})(1 + \frac{1}{2} var_{t+i}(Y_{t+i+1})) + \\
&+ (1-\pi^e)\exp(\overline{Z})(1 + \frac{1}{2} var_{t+i}(Z_{t+i+1})).
\end{aligned}
$$

Finally, a first-order expansion around zero provides

$$
\begin{aligned}
1 &\approx \pi^e \pi^p (1 + \overline{X} + \frac{1}{2} var_{t+i}(X_{t+i+1})) + \\
&+ \pi^e \pi^s (1 + \overline{Y} + \frac{1}{2} var_{t+i}(Y_{t+i+1})) + \\
&+ (1-\pi^e)(1 + \overline{Z} + \frac{1}{2} var_{t+i}(Z_{t+i+1}))
\end{aligned}
$$

or,

$$
\begin{aligned}
0 &= \sum_{j=p,s} \pi^e \pi^j (\log\beta - \gamma E_{t+i}(c^{ej}_{t+i+1} - c^{es}_{t+i}) + E_{t+i}(r_{i,t+i+1}) + \frac{1}{2} var_t(r_{i,t++i1} - \gamma(c^{ej}_{t+i+1} - c^{es}_{t+i})) + \\
&+ (1-\pi^e)(\log\beta^r - \gamma E_{t+i}(c^{r}_{t+i+1} - c^{es}_{t+i}) + E_{t+i}(r_{i,t+i+1}) + \frac{1}{2} var_t(r_{i,t++i1} - \gamma(c^{r}_{t+i+1} - c^{es}_{t+i})).
\end{aligned}
$$

- The Euler equation for the case where the agent is promoted is

$$
1 = E_t \left\{ \left[ \pi^e \beta \left( \frac{C^{ep}_{t+i+1}}{C^{ep}_{t+i}} \right)^{1-\gamma} + (1-\pi^e)\beta^r \left( \frac{C^{r}_{t+i}}{C^{ep}_{t+i}} \right)^{-\gamma} \right] (1 + R_{i,t+i+1}) \right\}.
$$

Taking the logarithm and exponential in the equation between square brackets we have

$$\begin{aligned} 1 &= \pi^e E_{t+i}[\exp(\log\beta - \gamma(c^{ep}_{t+i+1} - c^{ep}_{t+i}) + r_{i,t+i+1})] + \\ &+(1-\pi^e)E_{t+i}[\exp(\log\beta^r - \gamma(c^{r}_{t+i+1} - c^{ep}_{t+i}) + r_{i,t+i+1}) \end{aligned}$$

$$1 = \pi^e E_{t+i}[\exp(X_{t+i+1})] + (1-\pi^e)E_{t+i}[\exp(Y_{t+i+1})].$$

Making a second-order expansion of exp(X$_{t+i+1}$) and $\exp(Y_{t+i+1})$ around their respective expected values $\overline{X} = E(\exp(X_{t+i+1}))$ and $\overline{Y} = E(\exp(Y_{t+i+1}))$ we may write

$$\begin{aligned} 1 &\approx \pi^e E_{t+i}[\exp(\overline{X})(1 + (X_{t+i+1} - \overline{X}) + \frac{1}{2}(X_{t+i+1} - \overline{X})^2)] + \\ &+(1-\pi^e)E_{t+i}[\exp(\overline{Y})(1 + (Y_{t+i+1} - \overline{Y}) + (Y_{t+i+1} - \overline{Y})^2)] \end{aligned}$$

$$1 \approx \pi^e \exp(\overline{X})(1 + \frac{1}{2}var_{t+i}(X_{t+i+1})) + (1-\pi^e)\exp(\overline{Y})(1 + \frac{1}{2}var_{t+i}(Y_{t+i+1})).$$

Finally, making a first-order around zero, we obtain

$$1 \approx \pi^e(1 + \overline{X} + \frac{1}{2}var_{t+i}(X_{t+i+1})) + (1-\pi^e)(1 + \overline{Y} + \frac{1}{2}var_{t+i}(Y_{t+i+1}))$$

or

$$
\begin{aligned}
0 &= \pi^e(\log\beta - \gamma E_{t+i}(c^{ep}_{t+i+1} - c^{ep}_{t+i}) + E_{t+i}(r_{i,t+i+1}) + \frac{1}{2}var_t(r_{i,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{ep}_{t+i})) + \\
&\quad +(1-\pi^e)(\log\beta^r - \gamma E_{t+i}(c^{r}_{t+i+1} - c^{ep}_{t+i}) + E_{t+i}(r_{i,t+i+1}) + \frac{1}{2}var_t(r_{i,t++i1} - \gamma(c^{r}_{t+i+1} - c^{ep}_{t+i})).
\end{aligned}
$$

- The Euler equation for the case where the agent is retired is

$$1 = E_t\left[\beta^r\left(\frac{C^r_{t+i+1}}{C^r_{t+i}}\right)^{-\gamma}(1+R_{i,t+i+1})\right].$$

Taking the logarithm and exponential in the equation between square brackets we have

$$1 = E_{t+i}[\exp(\log\beta^r - \gamma(c^r_{t+i+1} - c^r_{t+i}) + r_{i,t+i+1}).$$

Expanding $\exp(X_{t+i+1})$ up to second order around its expected value $\overline{X} = E(\exp(X_{t+i+1}))$ we may write

$$1 \approx E_{t+i}[\exp(\overline{X})(1 + (X_{t+i+1} - \overline{X}) + \frac{1}{2}(X_{t+i+1} - \overline{X})^2)]$$

$$1 \approx \exp(\overline{X})(1 + \frac{1}{2}var_{t+i}(X_{t+i+1})).$$

Finally, making a first-order expansion around zero,

$$1 \approx (1 + \overline{X} + \frac{1}{2}var_{t+i}(X_{t+i+1}))$$

or

$$0 = \log \beta^r - \gamma E_{t+i}(c^r_{t+i+1} - c^r_{t+i}) + E_{t+i}(r_{i,t+i+1}) + \frac{1}{2} var_t \left[ r_{i,t++i1} - \gamma(c^r_{t+i+1} - c^r_{t+i}) \right].$$

# B The Investor's Optimal Rules for Each State

## B.1 Proof of Proposition 1

To prove *Proposition 1* we first guess that optimal policies take the form $\alpha^r_{t+i} = \alpha^r$ and $c^r_{t+i} = b^r_0 + b^r_1 w_{t+i}$. We next show that this guess verifies the log-linear Euler equation 27 and the log budget constraint 23, provided that $\alpha^r, b^r_0$ and $b^r_1$ satisfy three equations whose coefficients are functions of the primitive parameters that define the preference and stochastic structure of the problem.

We first try the following functional form for the optimal rules in the state where the agent is retired.

$$c^r_{t+i} = b^r_0 + b^r_1 w^r_{t+i}$$

$$\alpha^r_{t+i} = \alpha^r.$$

With this functional form we obtain the optimal rule for the portfolio in the state where the agent is retired. First, we subtract equation (27) for $i = f$ from the same equation (27) where $i$ is the risky asset:

$$E_{t+i}(r_{t+i+1}) - r_f + \frac{1}{2}\sigma^2_u = \gamma cov_{t+i}(r_{t+i+1}, c^r_{t+i+1} - c^r_{t+i}).$$

However, we must obtain the value of $cov_{t+i}(r_{t+i+1}, c^r_{t+i+1} - c^r_{t+i})$.

Equations (28) and (23), together with the trivial equality $c^r_{t+i+1} - c^r_{t+i} = b^r_1(w^r_{t+i+1} - w^r_{t+i})$ imply that

$$
\begin{aligned}
cov_{t+i}(r_{t+i+1}, c^r_{t+i+1} - c^r_{t+i}) &= cov_{t+i}(r_{t+i+1}, b^r_1(w^r_{t+i+1} - w^r_{t+i})) \\
&= cov_{t+i}(r_{t+i+1}, b^r_1(k^r - \rho^r_c(c_{t+i} - w_{t+i}) + r^r_{p,t+i+1})) \\
&= cov_{t+i}(r_{t+i+1}, b^r_1 r^r_{p,t+i+1}) \\
&= b^r_1 \alpha^r \sigma^2_u.
\end{aligned}
$$

Therefore,

$$E_{t+i}(r_{t+i+1}) - r_f + \frac{1}{2}\sigma^2_u = \gamma\alpha b^r_1 \sigma^2_u$$

$$\alpha^r = \frac{\mu + \frac{\sigma^2_u}{2}}{\gamma b^r_1 \sigma^2_u}.$$

Deriving the optimal consumption rule for retired agents through equation (27) with $i = p$, we obtain the logarithm of the expected consumption growth as

$$E_{t+i}(c^r_{t+i+1} - c^r_{t+i}) = \frac{1}{\gamma}(\log\beta^r + E_{t+i}(r_{p,t+i+1}) + \frac{1}{2}var_{t+i}(r_{p,t+i+1} - \gamma(c^r_{t+i+1} - c^r_{t+i})) = \Psi^r$$

where

$$\begin{aligned} var_{t+i}(r_{p,t++i1} - \gamma(c^r_{t+i+1} - c^r_{t+i})) &= var_{t+i}(r_{p,t+i+1} - \gamma(b^r_1(w^r_{t+i+1} - w^r_{t+i})) \\ &= var_{t+i}(r_{p,t+i+1} - \gamma(b^r_1 r^r_{p,t+i+1})) \\ &= (1-\gamma b^r_1)^2 var_{t+i}(r_{p,t+i+1}). \end{aligned}$$

Using the trivial equality above, together with equation (23) and (28), we have

$$\begin{aligned} E_{t+i}(c^r_{t+i+1} - c^r_{t+i}) &= b^r_1 E_{t+i}(w^r_{t+i+1} - w^r_{t+i}) = \Psi^r \\ &= b^r_1 E_{t+i}(r^r_{p,t+i+1}) - b^r_1\rho^r_c b^r_0 + b^r_1 k^r + b^r_1 \rho^r_c (1-b^r_1) w_{t+i} \end{aligned}$$

$$\frac{1}{\gamma}(\log\beta^r + E_{t+i}(r_{p,t+i+1}) + \frac{1}{2}(1-\gamma b^r_1)^2 var_{t+i}(r_{p,t+i+1})$$

$$= b^r_1 E_{t+i}(r^r_{p,t+i+1}) - b^r_1\rho^r_c b^r_0 + b^r_1 k^r + b^r_1\rho^r_c(1-b^r_1)w_{t+i}.$$

Identifying the coefficients on both sides of the equation, we have

$$b^r_1 = 1$$

and

$$b^r_0 = -\left(\frac{1}{b^r_1\rho^r_c}\right)\left[\left(\frac{1}{\gamma} - b^r_1\right)E[r_{p,t+i+1}] + \frac{1}{\gamma}\log\beta + \frac{1}{2\gamma}(1-\gamma b^r_1)^2 var_{t+i}(r_{p,t+i+1}) - b^r_1 k^r\right].$$

## B.2 Proof of Proposition 2

We first try the following functional form for the optimal rules in the state where the agent is promoted:

$$c_{t+i}^{ep} - \tau_{t+i} = b_0^{ep} + b_1^{ep}(w_{t+i}^{ep} - \tau_{t+i})$$

$$\alpha_{t+i}^{ep} = \alpha^{ep}.$$

With this functional form we obtain the optimal rule for the portfolio in the state where the agent is employed. First, we subtract equation (26) for $i = f$ of the same equation (26) where $i$ denotes the risky asset:

$$E_{t+i}(r_{t+i+1}) - r_f + \frac{1}{2}\sigma_u^2 = \gamma[\pi^e cov_{t+i}\left(r_{t+i+1}, c_{t+i+1}^{ep} - c_{t+i}^{ep}\right) + (1-\pi^e) cov_{t+i}(r_{t+i+1}, c_{t+i+1}^{r} - c_{t+i}^{ep})].$$

As in the retired case, we should obtain those covariances.

By the guessed functional form, the log of the constraint, and a trivial inequality we have:

$$
\begin{aligned}
cov_{t+i}(r_{t+i+1}, c^{ep}_{t+i+1} - c^{ep}_{t+i}) &= cov_{t+i}(r_{t+i+1}, (c^{ep}_{t+i+1} - \tau_{t+i+1}) - \\
&\quad -(c^{ep}_{t+i} - \tau_{t+i}) + (\tau_{t+i+1} - \tau_{t+i})) \\
&= cov_{t+i}(r_{t+i+1}, b^{ep}_1(w^{ep}_{t+i+1} - \tau_{t+i+1}) + \Delta\tau_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, b^{ep}_1(-\Delta\tau_{t+i+1} + r_{p,t+i+1}) + \Delta\tau_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, (1 - b^{ep}_1)\Delta\tau_{t+i+1} + b^{ep}_1 \alpha r_{t+i+1}) \\
&= (1 - b^{ep}_1) z\sigma_{\xi u} + \alpha b^{ep}_1 \sigma^2_u
\end{aligned}
$$

and

$$
\begin{aligned}
cov_{t+i}(r_{t+i+1}, c^{r}_{t+i+1} - c^{ep}_{t+i}) &= cov_{t+i}(r_{t+i+1}, (c^{r}_{t+i+1} - \tau_{t+i+1}) - \\
&\quad -(c^{e\tau}_{t+i} - \tau_{t+i}) + (\tau_{t+i+1} - \tau_{t+i})) \\
&= cov_{t+i}(r_{t+i+1}, b^{r}_1(w^{r}_{t+i+1} - \tau_{t+i+1}) + \Delta\tau_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, b^{r}_1(-\Delta\tau_{t+i+1} + r_{p,t+i+1}) + \Delta\tau_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, (1 - b^{r}_1)c\Delta\tau_{t+i+1} + b^{r}_1 \alpha r_{t+i+1}) \\
&= (1 - b^{r}_1) z\sigma_{\xi u} + \alpha b^{r}_1 \sigma^2_u \\
&= \alpha b^{r}_1 \sigma^2_u.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
E_{t+i}(r_{t+i+1}) - r_f + \frac{1}{2}\sigma_u^2 &= \gamma[\pi^e cov_{t+i}(r_{t+i+1}, c^{ep}_{t+i+1} - c^{ep}_{t+i}) + (1-\pi^e) cov_{t+i}(r_{t+i+1}, c^{r}_{t+i+1} - c^{ep}_{t+i})] \\
&= \gamma[\pi^e((1-b^{ep})z\sigma_{\xi u} + \alpha b_1^{ep}\sigma_u^2) + (1-\pi^e)(\alpha b_1^r \sigma_u^2)] \\
&= \alpha\gamma b_1^c \sigma_u^2 + \gamma\pi^e(1-b_1^{ep})\sigma_{\xi u}
\end{aligned}
$$

$$
\begin{aligned}
\alpha^{ep} &= \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^c \sigma_u^2} - \frac{\pi^e(1-b_1^{ep})z\sigma_{\xi u}}{b_1^c\sigma_u^2} \\
\alpha^{ep} &= \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^c \sigma_u^2} - \frac{\pi^e(1-b_1^{ep})\sigma_{\xi u}}{b_1^c\sigma_u^2} - \frac{\pi^e(1-b_1^{ep})(z-1)\sigma_{\xi u}}{b_1^c\sigma_u^2}.
\end{aligned}
$$

Deriving the optimal consumption rule for promoted investor, by equation (26) with $i = p$ we get the logarithm of expected consumption growth:

$$
\begin{aligned}
\pi^e E_{t+i}(c^{ep}_{t+i+1} - c^{ep}_{t+i}) + (1-\pi^e)E_{t+i}(c^{r}_{t+i+1} - c^{ep}_{t+i}) &= \frac{1}{\gamma}[\pi^e(\log\beta + E_{t+i}(r_{p,t+i+1}) \\
&\quad + \frac{1}{2}var_{t+i}(r_{p,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{ep}_{t+i})) \\
&\quad + (1-\pi^e)(\log\beta^r + E_{t+i}(r_{p,t+i+1}) \\
&\quad + \frac{1}{2}var_{t+i}(r_{p,t++i1} - \gamma(c^{r}_{t+i+1} - c^{ep}_{t+i}))] \\
&= \Psi^{ep}
\end{aligned}
$$

where

$$
V^{ep} = \pi^e var_{t+i}(r_{p,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{ep}_{t+i})) + (1-\pi^e) var_{t+i}(r_{p,t++i1} - \gamma(c^{r}_{t+i+1} - c^{ep}_{t+i}))
$$

$$\begin{aligned}
var_{t+i}(r_{p,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{ep}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma((c^{ep}_{t+i+1} - \tau_{t+i}) + \Delta\tau_{t+i+1})) \\
&= var_{t+i}(r_{p,t++i1} - \gamma(b^{ep}_1(w^{ep}_{t+i+1} - \tau_{t+i}) + \Delta\tau_{t+i+1})) \\
var_{t+i}(r_{p,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{ep}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma(b^{ep}_1(r_{p,t+i+1} - \Delta\tau_{t+i+1}) + \Delta\tau_{t+i+1})) \\
&= var_{t+i}((1-\gamma b^{ep}_1)r_{p,t+i+1} - \gamma(1-b^{ep}_1)\Delta\tau_{t+i+1}) \\
&= (1-\gamma b^{ep}_1)^2 var_{t+i}(r_{p,t+i+1}) + \gamma^2 z^2 (1-b^{ep}_1)^2 var_{t+i}(\Delta l_{t+i+1}) - \\
&\quad -2\gamma z(1-\gamma b^{ep}_1)(1-b^{ep}_1)cov_{t+i}(r_{p,t+i+1}, \Delta l_{t+i+1})
\end{aligned}$$

and

$$\begin{aligned}
var_{t+i}(r_{p,t++i1} - \gamma(c^{r}_{t+i+1} - c^{ep}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma((c^{r}_{t+i+1} - \tau_{t+i}) + \Delta\tau_{t+i+1}) \\
&= var_{t+i}(r_{p,t++i1} - \gamma(b^{r}_1(w^{r}_{t+i+1} - \tau_{t+i}) + \Delta\tau_{t+i+1}) \\
var_{t+i}(r_{p,t++i1} - \gamma(c^{r}_{t+i+1} - c^{ep}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma(b^{r}_1(r_{p,t+i+1} - \Delta\tau_{t+i+1}) + \Delta\tau_{t+i+1}) \\
&= var_{t+i}((1-\gamma b^{r}_1)r_{p,t+i+1} - \gamma(1-b^{r}_1)\Delta\tau_{t+i+1}) \\
&= (1-\gamma b^{r}_1)^2 var_{t+i}(r_{p,t+i+1}).
\end{aligned}$$

Then:

$$\begin{aligned}
V^{ep} &= \pi^e(1-\gamma b^{ep}_1)^2 var_{t+i}(r_{p,t+i+1}) + (1-\pi^e)(1-\gamma b^{r}_1)^2 var_{t+i}(r_{p,t+i+1}) + \\
&\quad \pi^e\gamma^2 z^2(1-b^{ep}_1)^2 var_{t+i}(\Delta l_{t+i+1}) - 2\gamma z\pi^e(1-\gamma b^{ep}_1)(1-b^{ep}_1)cov_{t+i}(r_{p,t+i+1}, \Delta l_{t+i+1}).
\end{aligned}$$

Using the equality above, together with equation (22) and (30) we have

$$\pi^e E_{t+i}(c^{ep}_{t+i+1} - c^{ep}_{t+i}) + (1-\pi^e)E_{t+i}(c^r_{t+i+1} - c^{ep}_{t+i}) = \pi^e E_{t+i}(c^{ep}_{t+i+1} - \tau_{t+i+1}) + (1-\pi^e)E_{t+i}(c^r_{t+i+1} -$$

$$= (c^{ep}_{t+i} - \tau_{t+i}) - zg + \Psi^{ep}$$

$$\pi^e E_{t+i}(b^{ep}_0 + b^{ep}_1(w^{ep}_{t+i+1} - \tau_{t+i+1})) + (1-\pi^e)E_{t+i}(b^r_0 + b^r_1(w^r_{t+i+1} - \tau_{t+i+1}))$$

$$= (c^{ep}_{t+i} - \tau_{t+i}) - zg + \Psi^{ep}$$

$$\pi^e E_{t+i}(b^{ep}_0 + b^{ep}_1(w^{ep}_{t+i+1} - \tau_{t+i+1})) + (1-\pi^e)E_{t+i}(b^r_0 + b^r_1(w^r_{t+i+1} - \tau_{t+i+1}))$$

$$= b^{ep}_0 + b^{ep}_1(w^{ep}_{t+i} - \tau_{t+i}) - zg + \Psi^{ep}$$

$$b^c_0 + b^c_1 E_{t+i}(w^{ep}_{t+i+1} - \tau_{t+i+1}) = b^{ep}_0 + b^{ep}_1(w^{ep}_{t+i} - \tau_{t+i}) - zg + \Psi^{ep}$$

$$b^c_1(\rho^{ep}_w - \rho^{ep}_c b^{ep}_1)(w_{t+i} - \tau_{t+i}) + b^c_0 + b^c_1(k^{ep} - \rho^{ep}_c b^{ep}_0 - zg + E_{t+i}(r_{p,t+i+1}))$$

$$= b^{ep}_0 + b^{ep}_1(w^{ep}_{t+i} - \tau_{t+i}) - zg + \Psi^{ep}.$$

Identifying the coefficients on both sides of the equation yields

$$b^c_1(\rho^{ep}_w - \rho^{ep}_c b^{ep}_1) = b^{ep}_1$$

$$b_0^c + b_1^c(k^{ep} - \rho_c^{ep} b_0^{ep} - zg + E_{t+i}(r_{p,t+i+1})) = b_0^{ep} - zg + \Psi^{ep}.$$

There are two equations and two unknowns. From the first equation we obtain directly

$$\pi^e \rho_c^{ep} (b_1^{ep})^2 + (1 - \pi^e \rho_w^{ep} + (1 - \pi^e) b_1^r \rho_c^{ep}) b_1^{ep} - (1 - \pi^e) b_1^r \rho_w^{ep} = 0.$$

The expression for $b_0^{ep}$ is given by:

$$\begin{aligned} b_0^{ep} &= -\frac{1}{[(1-\pi^e) + \rho_c^{ep} b_1^c]} [\left(\frac{1}{\gamma} - b_1^c\right) E_{t+i}(r_{p,t+i+1}) + \frac{1}{\gamma}(\pi^e \log \beta + (1-\pi^e) \log \beta^r) \\ &\quad + \frac{1}{2\gamma} V^{ep} + \pi^e (b_1^{ep} - 1) zg - b_1^c k^{ep} - (1-\pi^e) b_0^r]. \end{aligned}$$

Checking if $1 > b_1^{ep} > 0$ :

$$(b_1^{ep})^2 + \frac{(1 - \pi^e \rho_w^{ep} + (1-\pi^e) b_1^r \rho_c^{ep})}{\pi^e \rho_c^{ep}} b_1^{ep} - \frac{(1-\pi^e) b_1^r \rho_w^{ep}}{\pi^e \rho_c^{ep}} = 0.$$

The discriminant of this quadratic equation is

$$\left(\frac{(1 - \pi^e \rho_w^{ep} + (1-\pi^e) b_1^r \rho_c^{ep})}{\pi^e \rho_c^{ep}}\right)^2 + 4\left(\frac{(1-\pi^e) b_1^r \rho_w^{ep}}{\pi^e \rho_c^{ep}}\right) \geq 0$$

since the first term is quadratic and the second term has only positive coefficients. Therefore, the roots are real.

By the last term of the quadratic equation, because it is negative we know that there exists one positive root and another negative. Discarding the negative root because it would imply a consumption level decreasing in the wealth for all levels of working income, making the agent better off with less wealth, so $b_1^{ep} > 0$.

To prove that $b_1^{ep} < 1$, we will show that $b_1^{ep} \geq 1$ is inconsistent with the fact of investor saving $(W_{t+i} + L_{t+i} + T_{t+i} - C_{t+i}) > 0$ during the optimal path. If $b_1^{ep} \geq 1$ then:

$$b_1^{ep} = \frac{-[1 - \pi^e \rho_w^{ep} + (1 - \pi^e) b_1^r \rho_c^{ep}] + \sqrt{(1 - \pi^e \rho_w^{ep} + (1 - \pi^e) b_1^r \rho_c^{ep})^2 + 4\pi^e \rho_c^{ep} (1 - \pi^e) b_1^r \rho_w^{ep}}}{2\pi^e \rho_c^{ep}} \geq 1.$$

Taking the square of both sides and solving we have

$$b_1^r(\rho_w^{ep} - \rho_c^{ep}) - \pi^e b_1^r(\rho_w^{ep} - \rho_c^{ep}) + \pi^e(\rho_w^{ep} - \rho_c^{ep}) \geq 1.$$

Because we know that $b_1^r = 1$,

$$(\rho_w^{ep} - \rho_c^{ep}) \geq 1$$

$$\exp E\left(w_{t+i}^{ep} - l_{t+i}\right) - \exp E\left(c_{t+i}^{ep} - l_{t+i}\right) \geq 1 + \exp E\left(w_{t+i}^{ep} - l_{t+i}\right) - \exp E\left(c_{t+i}^{ep} - l_{t+i}\right),$$

the left hand side of this inequality equals $0$ and the right hand side equals $1$, leading to a contradiction. Therefore, we must have $b_1^{ep} < 1$.

## B.3 Proof of Proposition 4

We first try the following functional form for the optimal rules in the state where the agent is employed spending in education:

$$c_{t+i}^{es} - l_{t+i} = b_0^{es} + b_1^{es}(w_{t+i}^{es} - l_{t+i})$$

$$x_{t+i} - l_{t+i} = b_2^{es} + b_3^{es}(w_{t+i}^{es} - l_{t+i})$$

$$\alpha_{t+i}^{es} = \alpha^{es}.$$

With this functional form we obtain the optimal rule for the portfolio in the state where the agent is employed. First, we subtract equation (24) for $i = f$ of the same equation (24) where $i$ denotes the risky asset:

$$\begin{aligned} E_{t+i}(r_{t+i+1}) - r_f + \frac{1}{2}\sigma_u^2 &= \gamma[\pi^e(\pi^p cov_{t+i}(r_{t+i+1}, c_{t+i+1}^{ep} - c_{t+i}^{es}) + \pi^s cov_{t+i}(r_{t+i+1}, c_{t+i+1}^{es} - c_{t+i}^{es})) \\ &\quad +(1-\pi^e)cov_{t+i}(r_{t+i+1}, c_{t+i+1}^{r} - c_{t+i}^{es})]. \end{aligned}$$

As in the cases above, we should obtain those covariances.

By the guessed functional form, by the log of the constraint, and by trivial inequality we have:

$$\begin{aligned} cov_{t+i}(r_{t+i+1}, c_{t+i+1}^{es} - c_{t+i}^{es}) &= cov_{t+i}(r_{t+i+1}, (c_{t+i+1}^{es} - l_{t+i+1}) - \\ &\quad -(c_{t+i}^{es} - l_{t+i}) + (l_{t+i+1} - l_{t+i})) \\ &= cov_{t+i}(r_{t+i+1}, b_1^{es}(w_{t+i+1}^{es} - l_{t+i+1}) + \Delta l_{t+i+1}) \\ &= cov_{t+i}(r_{t+i+1}, b_1^{es}(-\Delta l_{t+i+1} + r_{p,t+i+1}) + \Delta l_{t+i+1}) \\ &= cov_{t+i}(r_{t+i+1}, (1-b_1^{es})\Delta l_{t+i+1} + b_1^{es}\alpha r_{t+i+1}) \\ &= (1-b_1^{es})\sigma_{\xi u} + \alpha b_1^{es}\sigma_u^2 \end{aligned}$$

$$
\begin{aligned}
cov_{t+i}(r_{t+i+1}, c^{ep}_{t+i+1} - c^{es}_{t+i}) &= cov_{t+i}(r_{t+i+1}, (c^{ep}_{t+i+1} - \tau_{t+i+1}) - \\
&\quad -(c^{es}_{t+i} - \tau_{t+i}) + (\tau_{t+i+1} - \tau_{t+i})) \\
&= cov_{t+i}(r_{t+i+1}, b^{ep}_1(w^{ep}_{t+i+1} - \tau_{t+i+1}) + \Delta\tau_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, b^{ep}_1(-\Delta\tau_{t+i+1} + r_{p,t+i+1}) + \Delta\tau_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, (1 - b^{ep}_1)z\Delta l_{t+i+1} + b^{ep}_1\alpha r_{t+i+1}) \\
&= z(1 - b^{ep}_1)\sigma_{\xi u} + \alpha b^{ep}_1\sigma^2_u
\end{aligned}
$$

$$
\begin{aligned}
cov_{t+i}(r_{t+i+1}, c^{r}_{t+i+1} - c^{es}_{t+i}) &= cov_{t+i}(r_{t+i+1}, (c^{r}_{t+i+1} - l_{t+i+1}) - \\
&\quad -(c^{r}_{t+i} - l_{t+i}) + (l_{t+i+1} - l_{t+i})) \\
&= cov_{t+i}(r_{t+i+1}, b^{r}_1(w^{r}_{t+i+1} - l_{t+i+1}) + \Delta l_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, b^{r}_1(-\Delta l_{t+i+1} + r_{p,t+i+1}) + \Delta l_{t+i+1}) \\
&= cov_{t+i}(r_{t+i+1}, (1 - b^{r}_1)\Delta l_{t+i+1} + b^{el}_1\alpha r_{t+i+1}) \\
&= (1 - b^{r}_1)c\sigma_{\xi u} + \alpha b^{r}_1\sigma^2_u \\
&= \alpha b^{r}_1\sigma^2_u.
\end{aligned}
$$

Then

$$
\begin{aligned}
E_{t+i}(r_{t+i+1}) - r_f + \frac{1}{2}\sigma^2_u &= \gamma[\pi^e(\pi^p((1 - b^{ep}_1)z\sigma_{\xi u} + \alpha b^{ep}_1\sigma^2_u) \\
&\quad +\pi^s((1 - b^{es}_1)\sigma_{\xi u} + \alpha b^{es}_1\sigma^2_u)) + (1 - \pi^e)\alpha b^{r}_1\sigma^2_u].
\end{aligned}
$$

Rearranging, we have

$$\alpha^{es} = \frac{\mu + \frac{\sigma_u^2}{2}}{\gamma b_1^{cc}\sigma_u^2} - \frac{\pi^e\pi^s(1-b_1^{es})\sigma_{\xi u}}{b_1^{cc}\sigma_u^2} - \frac{\pi^e\pi^p(1-b_1^{ep})z\sigma_{\xi u}}{b_1^{cc}\sigma_u^2}$$

where $b_1^{cc} = \pi^e(\pi^p b_1^{ep} + \pi^s b_1^{es}) + (1-\pi^e)b_1^r$.

Deriving the optimal consumption rule for employed investor who educates himself, by equation (24) with $i = p$ we obtain the logarithm of the expected consumption growth:

$$\begin{aligned}
0 \;=\; & \sum_{j=l,s} \pi^e\pi^j(\log\beta - \gamma E_{t+i}(c_{t+i+1}^{ej} - c_{t+i}^{es}) + E_{t+i}(r_{p,t+i+1}) + \\
& + \frac{1}{2}var_{t+i}(r_{p,t+i+1} - \gamma(c_{t+i+1}^{ej} - c_{t+i}^{es}))) + (1-\pi^e)(\log\beta^r - \gamma E_{t+i}(c_{t+i+1}^r - c_{t+i}^{es}) + \\
& + E_{t+i}(r_{p,t+i+1}) + \frac{1}{2}var_{t+i}(r_{p,t+i+1} - \gamma(c_{t+i+1}^{ej} - c_{t+i}^{es})))
\end{aligned}$$

$$\begin{aligned}
& \pi^e\pi^p E_{t+i}(c_{t+i+1}^{ep} - c_{t+i}^{es}) + \pi^e\pi^s E_{t+i}(c_{t+i+1}^{es} - c_{t+i}^{es}) + (1-\pi^e)E_{t+i}(c_{t+i+1}^r - c_{t+i}^{es}) \\
= \;& \frac{1}{\gamma}(E_{t+i}(r_{p,t+i+1}) + \frac{1}{2}V^{es} + \pi^e\log\beta + (1-\pi^e)\log\beta^r) = \Psi^{es}.
\end{aligned}$$

Using the trivial inequalities we have

$$\begin{aligned}
& \pi^e\pi^p E_{t+i}(c_{t+i+1}^{ep} - c_{t+i}^{es}) + \pi^e\pi^s E_{t+i}(c_{t+i+1}^{es} - c_{t+i}^{es}) + (1-\pi^e)E_{t+i}(c_{t+i+1}^r - c_{t+i}^{es}) \qquad (35) \\
= \;& \pi^e\pi^p E_{t+i}(c_{t+i+1}^{ep} - l_{t+i+1}) + \pi^e\pi^s E_{t+i}(c_{t+i+1}^{es} - l_{t+i+1}) + (1-\pi^e)E_{t+i}(c_{t+i+1}^r - l_{t+i+1}) \\
= \;& (c_{t+i}^{ep} - l_{t+i}) + \Psi^{es} - g.
\end{aligned}$$

The covariances are given by

$$
\begin{aligned}
V^{es} &= \pi^e \pi^p var_{t+i}(r_{p,t+i+1} - \gamma(c^{ep}_{t+i+1} - c^{es}_{t+i})) + \pi^e \pi^s var_{t+i}(r_{p,t+i+1} - \gamma(c^{es}_{t+i+1} - c^{es}_{t+i})) + \\
&\quad + (1-\pi^e) var_{t+i}(r_{p,t+i+1} - \gamma(c^{r}_{t+i+1} - c^{es}_{t+i}))
\end{aligned}
$$

where

$$
\begin{aligned}
var_{t+i}(r_{p,t++i1} - \gamma(c^{es}_{t+i+1} - c^{es}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma((c^{es}_{t+i+1} - l_{t+i}) + \Delta l_{t+i+1}) \\
&= var_{t+i}(r_{p,t++i1} - \gamma(b^{es}_1(w^{es}_{t+i+1} - l_{t+i}) + \Delta l_{t+i+1}) \\
var_{t+i}(r_{p,t++i1} - \gamma(c^{es}_{t+i+1} - c^{es}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma(b^{es}_1(r_{p,t+i+1} - \Delta l_{t+i+1}) + \Delta l_{t+i+1}) \\
&= var_{t+i}((1-\gamma b^{es}_1) r_{p,t+i+1} - \gamma(1-b^{es}_1)\Delta l_{t+i+1}) \\
&= (1-\gamma b^{es}_1)^2 var_{t+i}(r_{p,t+i+1}) + \gamma^2 (1-b^{es}_1)^2 var_{t+i}(\Delta l_{t+i+1}) - \\
&\quad -2\gamma(1-\gamma b^{es}_1)(1-b^{es}_1) cov_{t+i}(r_{p,t+i+1}, \Delta l_{t+i+1})
\end{aligned}
$$

$$
\begin{aligned}
var_{t+i}(r_{p,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{es}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma((c^{ep}_{t+i+1} - \tau_{t+i}) + \Delta \tau_{t+i+1}) \\
&= var_{t+i}(r_{p,t++i1} - \gamma(b^{ep}_1(w^{ep}_{t+i+1} - \tau_{t+i}) + \Delta \tau_{t+i+1}) \\
var_{t+i}(r_{p,t++i1} - \gamma(c^{ep}_{t+i+1} - c^{es}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma(b^{ep}_1(r_{p,t+i+1} - \Delta \tau_{t+i+1}) + \Delta \tau_{t+i+1}) \\
&= var_{t+i}((1-\gamma b^{ep}_1) r_{p,t+i+1} - \gamma(1-b^{ep}_1)\Delta \tau_{t+i+1}) \\
&= (1-\gamma b^{ep}_1)^2 var_{t+i}(r_{p,t+i+1}) + \gamma^2 z^2 (1-b^{ep}_1)^2 var_{t+i}(\Delta l_{t+i+1}) - \\
&\quad -2\gamma z(1-\gamma b^{ep}_1)(1-b^{ep}_1) cov_{t+i}(r_{p,t+i+1}, \Delta l_{t+i+1})
\end{aligned}
$$

$$
\begin{aligned}
var_{t+i}(r_{p,t++i1} - \gamma(c^{r}_{t+i+1} - c^{es}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma((c^{r}_{t+i+1} - l_{t+i}) + \Delta l_{t+i+1}) \\
&= var_{t+i}(r_{p,t++i1} - \gamma(b^{r}_{1}(w^{r}_{t+i+1} - l_{t+i}) + \Delta l_{t+i+1}) \\
var_{t+i}(r_{p,t++i1} - \gamma(c^{r}_{t+i+1} - c^{es}_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma(b^{r}_{1}(r_{p,t+i+1} - \Delta l_{t+i+1}) + \Delta l_{t+i+1}) \\
&= var_{t+i}((1 - \gamma b^{r}_{1})r_{p,t+i+1} - \gamma(1 - b^{r}_{1})\Delta l_{t+i+1}) \\
&= (1 - \gamma b^{r}_{1})^{2} var_{t+i}(r_{p,t+i+1}) + \gamma^{2}(1 - b^{r}_{1})^{2} var_{t+i}(\Delta l_{t+i+1}) - \\
&\quad -2\gamma(1 - \gamma b^{r}_{1})(1 - b^{r}_{1}) cov_{t+i}(r_{p,t+i+1}, \Delta l_{t+i+1}).
\end{aligned}
$$

Substituting the equations of consumption (30), (32), and (28) in the equation (35) we have

$$
\begin{aligned}
& \pi^{e}\pi^{p}E_{t+i}(b^{ep}_{0} + b^{ep}_{1}(w_{t+i+1} - l_{t+i+1}) + (1 - b^{ep}_{1})(z - 1)l_{t+i+1}) + \\
& +\pi^{e}\pi^{s}E_{t+i}(b^{es}_{0} + b^{es}_{1}(w_{t+i+1} - l_{t+i+})) + (1 - \pi^{e})E_{t+i}(b^{r}_{0} + b^{r}_{1}(w_{t+i+1} - l_{t+i+1})) \\
= \quad & b^{es}_{0} + b^{es}_{1}(w_{t+i+1} - l_{t+i+1}) + \Psi^{es} - g
\end{aligned}
$$

$b^{cc}_{k} = \pi^{e}(\pi^{p}b^{ep}_{k} + \pi^{s}b^{es}_{k}) + (1 - \pi^{e})b^{r}_{k}$, for all k=0,1, then:

$$
\begin{aligned}
& b^{cc}_{0} + b^{cc}_{1}(w_{t+i+1} - l_{t+i+1}) + \pi^{e}\pi^{p}(1 - b^{ep}_{1})(\tau_{t+i+1} - l_{t+i+1}) \pm \pi^{e}\pi^{p}(1 - b^{ep}_{1})(l_{t+i} - l_{t+i}) \\
= \quad & b^{es}_{0} + b^{es}_{1}(w_{t+i+1} - l_{t+i+1}) + \Psi^{es} - g
\end{aligned}
$$

$$
\begin{aligned}
& b_0^{cc} + b_1^{cc}(w_{t+i+1} - l_{t+i+1}) + \pi^e \pi^p [(1 - b_1^{ep})(\tau_{t+i+1} - l_{t+i}) - (1 - b_1^{ep})(l_{t+i+1} - l_{t+i})] \\
= \; & b_0^{cc} + b_1^{cc}(w_{t+i+1} - l_{t+i+1}) + \pi^e \pi^p (1 - b_1^{ep})(z - 1)g \\
= \; & b_0^{es} + b_1^{es}(w_{t+i+1} - l_{t+i+1}) + \Psi^{es} - g
\end{aligned}
$$

we assume that $\tau_{t+i} = l_{t+i}$.

$$
\begin{aligned}
& b_1^{cc}(\rho_w^{es} - \rho_x^{es} b_3^{es} - \rho_c^{es} b_1^{es})(w_{t+i}^{es} - l_{t+i}) + b_0^{cc} + b_1^{cc}(k^{es} - \rho_x^{es} b_2^{es} - \rho_c^{es} b_0^{es} - g + E_{t+i}(r_{p,t+i+1})) + \\
& + \pi^e \pi^p (1 - b_1^{ep})(z - 1)g \\
= \; & b_0^{es} + b_1^{es}(w_{t+i}^{es} - l_{t+i}) + \Psi^{es} - g.
\end{aligned}
$$

Identifying the coefficients $b_0^{es}, b_1^{es}$ by equations

$$
b_1^{cc}(\rho_w^{es} - \rho_x^{es} b_3^{es} - \rho_c^{es} b_1^{es}) = b_1^{es}
$$

$$
\begin{aligned}
& b_0^{cc} + b_1^{cc}(k^{es} - \rho_x^{es} b_2^{es} - \rho_c^{es} b_0^{es} - g + E_{t+i}(r_{p,t+i+1})) + \pi^e \pi^p (1 - b_1^{ep})(z - 1)g \\
= \; & \Psi^{es} - g + b_0^{es}.
\end{aligned}
$$

Solving recursively, because the first equation depends only on $b_1^{es}$, and the second on $b_0^{ep}, b_1^{ep}$, and manipulating the first equation we find the following quadratic form:

$$
\begin{aligned}
0 &= \pi^e\pi^s\rho_c^{es}(b_1^{es})^2 + (1-\pi^e\pi^s(\rho_w^{es}-\rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})b_1^{es} \\
&\quad - b_1^{ep}\pi^e\pi^p(\rho_w^{es}-\rho_x^{es}b_3^{es}) - (1-\pi^e)b_1^r(\rho_w^{es}-\rho_x^{es}b_3^{es})
\end{aligned}
$$

We have this quadratic form for $b_1^{es}$, a function of only the parameters and probabilities.

The expression for $b_0^{es}$ is given by

$$
\begin{aligned}
b_0^{es} &= -\frac{1}{[(1-\pi^e\pi^s)+b_1^{cc}\rho_c^{es}]}[\left(\frac{1}{\gamma}-b_1^{cc}\right)E_{t+i}(r_{p,t+i+1}) + \frac{1}{\gamma}(\pi^e\log\beta + (1-\pi^e)\log\beta^r) + \\
&\quad \frac{1}{2\gamma}V^{es} - b_1^{cc}(k^{es}-b_2^{es}\rho_x^{es}) + g(b_1^{cc}-1) - (\pi^e\pi^p b_0^{ep} + (1-\pi^e)b_0^r)) - \pi^e\pi^p(1-b_1^{ep})(z-1)g].
\end{aligned}
$$

Let us now characterize the roots of the quadratic equation of $b_1^{es}$.

$$
\begin{aligned}
0 &= (b_1^{es})^2 + \frac{(1-\pi^e\pi^s(\rho_w^{ep}-\rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})}{\pi^e\pi^s\rho_c^{es}}b_1^{es} - \\
&\quad -\frac{b_1^{ep}\pi^e\pi^p(\rho_w^{es}-\rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es}-\rho_x^{es}b_3^{es})}{\pi^e\pi^s\rho_c^{es}}.
\end{aligned}
$$

The discriminant of this quadratic equation is

$$
\begin{aligned}
0 &\leq \left(\frac{(1-\pi^e\pi^s(\rho_w^{ep}-\rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})}{\pi^e\pi^s\rho_c^{es}}\right)^2 + \\
&\quad +4\left(\frac{b_1^{ep}\pi^e\pi^p(\rho_w^{es}-\rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es}-\rho_x^{es}b_3^{es})}{\pi^e\pi^s\rho_c^{es}}\right)
\end{aligned}
$$

because the first term is quadratic and the second term have only positive coefficients. $b_1^{ep}, b_1^r, \rho_w^{es}, \rho_x^{es}, \rho_c^{es} > 0, 0 < \pi^e, \pi^p, \pi^s < 1$. Moreover $\rho_w^{es} > \rho_x^{es}b_3^{es}$, because the spending with education cannot be greater than the financial wealth, since the agent is a saver in this phase of life. Let us assume first that $0 < b_3^{es} < 1$. Then we will show that this is consistent with the result $1 > b_1^{es} > 0$.Therefore, the roots are real.

By the last term of the quadratic equation, because it is negative, we know that there exists one positive root and another negative. We do not consider the negative root because it would imply that the optimal level of consumption is decreasing in the wealth for all levels of working income, and the agent would then be better off with less wealth, so $b_1^{es} > 0$.

To prove that $b_1^{es} < 1$, we will show that $b_1^{es} \geq 1$ is not consistent with the fact of the investor having saved $(W_{t+i} + L_{t+i} + X_{t+i} - C_{t+i}) > 0$ during his optimal path. If $b_1^{es} \geq 1$ then

$$
\begin{aligned}
1 \quad \leq \quad b_1^{es} = & \frac{-(1-\pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})}{2\pi^e\pi^s\rho_c^{es}} + \\
& \frac{\sqrt{\begin{array}{c}(1-\pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})^2 + \\ +4\pi^e\pi^s\rho_c^{es}(b_1^{ep}\pi^e\pi^p(\rho_w^{es} - \rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es} - \rho_x^{es}b_3^{es}))\end{array}}}{2\pi^e\pi^s\rho_c^{es}}
\end{aligned}
$$

$$
\begin{aligned}
& 2\pi^e\pi^s\rho_c^{es} + (1-\pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es}) \\
\leq & \sqrt{\begin{array}{c}(1-\pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})^2 + \\ +4\pi^e\pi^s\rho_c^{es}(b_1^{ep}\pi^e\pi^p(\rho_w^{es} - \rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es} - \rho_x^{es}b_3^{es}))\end{array}}
\end{aligned}
$$

$$
\begin{aligned}
& (2\pi^e\pi^s\rho_c^{es} + (1 - \pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es}))^2 \\
\leq \; & (1 - \pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + \\
& +(1-\pi^e)b_1^r\rho_c^{es})^2 + 4\pi^e\pi^s\rho_c^{es}(b_1^{ep}\pi^e\pi^p(\rho_w^{es} - \rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es} - \rho_x^{es}b_3^{es})).
\end{aligned}
$$

Taking the square of both sides and solving, we have

$$
\begin{aligned}
& 4\pi^e\pi^s\rho_c^{es}(b_1^{ep}\pi^e\pi^p(\rho_w^{es} - \rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es} - \rho_x^{es}b_3^{es})) \\
\geq \; & 4\pi^{e^2}\pi^{s^2}\rho_c^{es^2} + 4\pi^e\pi^s\rho_c^{es}(1 - \pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})
\end{aligned}
$$

$$
\begin{aligned}
& (b_1^{ep}\pi^e\pi^p(\rho_w^{es} - \rho_x^{es}b_3^{es}) + (1-\pi^e)b_1^r(\rho_w^{es} - \rho_x^{es}b_3^{es})) \\
\geq \; & \pi^e\pi^s\rho_c^{es} + (1 - \pi^e\pi^s(\rho_w^{ep} - \rho_x^{es}b_3^{es}) + \pi^e\pi^p b_1^{ep}\rho_c^{es} + (1-\pi^e)b_1^r\rho_c^{es})
\end{aligned}
$$

$$
\begin{aligned}
1 - \pi^e\pi^s(\rho_w^{es} - \rho_c^{es} - \rho_x^{es}b_3^{es}) & \leq b_1^{ep}\pi^e\pi^p(\rho_w^{es} - \rho_c^{es} - \rho_x^{es}b_3^{es}) + (1-\pi^e)(\rho_w^{es} - \rho_c^{es} - \rho_x^{es}b_3^{es}) \\
1 & \leq (\rho_w^{es} - \rho_c^{es} - \rho_x^{es}b_3^{es})(\pi^e(\pi^s + b_1^{ep}\pi^p) + (1-\pi^e))
\end{aligned}
$$

where $1 + \rho_w^{es} - \rho_x^{es} - \rho_c^{es} > 0 \Rightarrow 1 > \rho_x^{es} + \rho_c^{es} - \rho_w^{es} > b_3^{es}\rho_x^{es} + \rho_c^{es} - \rho_w^{es}$, with $0 < b_1^{ep}, b_3^{es}, b_1^r, \pi^e, \pi^s, \pi^l < 1$.

For this inequality to be valid it would be necessary that

$$(\rho_w^{es}-\rho_c^{es}-\rho_x^{es}) \geq (\rho_w^{es}-\rho_c^{es}-\rho_x^{es}b_3^{es})(1) \geq (\rho_w^{es}-\rho_c^{es}-\rho_x^{es}b_3^{es})(\pi^e(\pi^s+b_1^{ep}\pi^p)+(1-\pi^e)) \geq 1$$

because $0 < b_1^{ep}, b_3^{es} < 1$, then

$$\rho_w^{es}-\rho_c^{es}-\rho_x^{es} \geq 1.$$

But this then implies that

$$\begin{aligned} & \exp E\left(w_{t+i}^{es}-l_{t+i}\right)-\exp E(x_{t+i}-l_{t+i})-\exp E\left(c_{t+i}^{es}-l_{t+i}\right) \\ \geq \quad & 1+\exp E\left(w_{t+i}^{es}-l_{t+i}\right)-\exp E(x_{t+i}-l_{t+i})-\exp E\left(c_{t+i}^{es}-l_{t+i}\right), \end{aligned}$$

the left hand side of this inequality equals $0$, and the right hand side equals $1$, leading to a contradiction.

Therefore, if even $\rho_w^{es}-\rho_c^{es}-\rho_x^{es} < 1$, we must have $(\rho_w^{es}-\rho_c^{es}-\rho_x^{es}b_3^{es})(\pi^e(\pi^s+b_1^{ep}\pi^p)+(1-\pi^e)) < 1$, because $(\rho_w^{es}-\rho_c^{es}-\rho_x^{es}) \geq (\rho_w^{es}-\rho_c^{es}-\rho_x^{es}b_3^{es})(\pi^e(\pi^s+b_1^{ep}\pi^p)+(1-\pi^e))$; which is a contradiction, so $b_1^{es} < 1 \Rightarrow 0 < b_1^{es} < 1$.

Deriving the optimal consumption rule for the spending with education upgrade, by equation (25) with $i = p$ we have

$$0 = \pi^e\pi^s E_{t+i}\{\log\beta-\gamma(c_{t+i+1}^{es}-x_{t+i})+r_{p,t+i+1}+\frac{1}{2}var_{t+i}(r_{p,t++i1}-\gamma(c_{t+i+1}^{es}-x_{t+i}))\}$$

$$0 = E_{t+i}\{\log\beta - \gamma(c^{es}_{t+i+1} - x_{t+i}) + r_{p,t+i+1} + \frac{1}{2}var_{t+i}(r_{p,t++i1} - \gamma(c^{es}_{t+i+1} - x_{t+i}))\}$$

$$\begin{aligned} E_{t+i}(c^{es}_{t+i+1} - x_{t+i}) &= \frac{1}{\gamma}(\log\beta + E_{t+i}(r_{p,t+i+1}) + \frac{1}{2}var_{t+i}(r_{p,t++i1} - \gamma(c^{es}_{t+i+1} - x_{t+i})) \\ &= \Psi^{es}_x \end{aligned}$$

where the variance is:

$$\begin{aligned} var_{t+i}(r_{p,t++i1} - \gamma(c^{es}_{t+i+1} - x_{t+i})) &= var_{t+i}(r_{p,t++i1} - \gamma(b^{es}_1(w^{es}_{t+i+1} - l_{t+i+1}) + \Delta l_{t+i+1})) \\ &= var_{t+i}(r_{p,t++i1} - \gamma(b^{es}_1(-\Delta l_{t+i+1} + r_{p,t+i+1}) + \Delta l_{t+i+1})) \\ &= var_{t+i}((1-\gamma b^{es}_1)r_{p,t+i+1} - \gamma(1-b^{es}_1)\Delta l_{t+i+1}) \\ &= (1-\gamma b^{es}_1)^2 var_{t+i}(r_{p,t+i+1}) + \gamma^2(1-b^{es}_1)^2 var_{t+i}(\Delta l_{t+i+1}) - \\ &\quad -2\gamma(1-b^{es}_1)(1-\gamma b^{es}_1)cov_{t+i}(r_{p,t+i+1}, \Delta l_{t+i+1}). \end{aligned}$$

Using the guess of the spent and consumption equation, the trivial inequality, and the logarithm of constraint equation we have

$$b^{es}_0 + b^{es}_1(w^{es}_{t+i+1} - l_{t+i+1}) = (x_{t+i} - l_{t+i}) - g + \Psi^{es}_x$$

$$
\begin{aligned}
& b_0^{es} + b_1^{es}(k^{es} + \rho_w^{es}(w_{t+i} - l_{t+i}) - \rho_x^{es}(b_2^{es} + b_3^{es}(w_{t+i} - l_{t+i})) - \\
& -\rho_c^{es}(b_0^{es} + b_1^{es}(w_{t+i} - l_{t+i}) - g + E_{t+i}(r_{p,t+i+1})) \\
= \;& b_2^{es} + b_3^{es}(w_{t+i} - l_{t+i}) - g + \Psi_x^{es}
\end{aligned}
$$

$$
\begin{aligned}
& b_1^{es}(\rho_w^{es} - \rho_x^{es} b_3^{es} - \rho_c^{es} b_1^{es})(w_{t+i} - l_{t+i}) + b_0^{es} + b_1^{es}(k^{es} - \rho_x^{es} b_2^{es} - \rho_c^{es} b_0^{es} - g + E_{t+i}(r_{p,t+i+1})) \\
= \;& b_3^{es}(w_{t+i} - l_{t+i}) + b_2^{es} - g + \Psi_x^{es}.
\end{aligned}
$$

Identifying the coefficients $b_2^{es}, b_3^{es}$ by equations:

$$
b_1^{es}(\rho_w^{es} - \rho_x^{es} b_3^{es} - \rho_c^{es} b_1^{es}) = b_3^{es}
$$

$$
b_0^{es} + b_1^{es}(k^{es} - \rho_x^{es} b_2^{es} - \rho_c^{es} b_0^{es} - g + E(r_{p,t+i+1})) = b_2^{es} - g + \Psi_x^{es}.
$$

Solving recursively, because the first equation depends only on $b_3^{es}$, and the second on $b_2^{ep}, b_3^{ep}$.

$$
b_3^{es}(1 + b_1^{es}\rho_x^{es}) = b_1^{es}(\rho_w^{es} - \rho_c^{es} b_1^{es})
$$

$$
b_3^{es} = \frac{b_1^{es}(\rho_w^{es} - \rho_c^{es} b_1^{es})}{(1 + b_1^{es}\rho_x^{es})} > 0.
$$

Now we shall show by contradiction that $b_3^{es} < 1$.

Assume that $b_3^{es} \geq 1$. In that case,

$$b_1^{es}(\rho_w^{es} - \rho_c^{es} b_1^{es}) \geq (1 + b_1^{es} \rho_x^{es})$$

$$(\rho_w^{es} - \rho_x^{es} - \rho_c^{es} b_1^{es}) \geq b_1^{es}(\rho_w^{es} - \rho_x^{es} - \rho_c^{es} b_1^{es}) \geq 1$$

because $0 < b_1^{es} < 1$, but

$$\frac{\exp E\left(w_{t+i}^{es} - l_{t+i}\right) - \exp E(x_{t+i} - l_{t+i}) - \exp E\left(c_{t+i}^{es} - l_{t+i}\right) b_1^{es}}{1 + \exp E\left(w_{t+i}^{es} - l_{t+i}\right) - \exp E(x_{t+i} - l_{t+i}) - \exp E\left(c_{t+i}^{es} - l_{t+i}\right)} \geq 1$$

reads

$$\begin{aligned} & \exp E\left(w_{t+i}^{es} - l_{t+i}\right) - \exp E(x_{t+i} - l_{t+i}) - \exp E\left(c_{t+i}^{es} - l_{t+i}\right) b_1^{es} \\ \geq \quad & 1 + \exp E\left(w_{t+i}^{es} - l_{t+i}\right) - \exp E(x_{t+i} - l_{t+i}) - \exp E\left(c_{t+i}^{es} - l_{t+i}\right). \end{aligned}$$

For $b_1^{es} \approx 1$, the left hand side of this inequality equals $0$ and the right hand side equals $1$, leading to a contradiction. For $b_1^{es} \approx 0$, we have

$$\exp E\left(c_{t+i}^{es} - l_{t+i}\right) \geq 1.$$

This is only possible if $c_{t+i}^{es} \geq l_{t+i}$, meaning that the investor is spending his savings in this phase of life, which is not compatible with the fact that he saved $(W_{t+i} + L_{t+i} - X_{t+i} - C_{t+i}) > 0$ during the optimal path. Hence, it follows that $0 < b_3^{es} < 1$.

Finding the value of the coefficient $b_2^{es}$ :

$$b_0^{es} + b_1^{es}(k^{es} - \rho_x^{es} b_2^{es} - \rho_c^{es} b_0^{es} - g + E_{t+i}(r_{p,t+i+1})) = b_2^{es} - g + \Psi_x^{es}$$

$$\begin{aligned} b_2^{es} &= -\frac{1}{(1+b_1^{es}\rho_x^{es})}[\left(\frac{1}{\gamma}-b_1^{es}\right)E_{t+i}(r_{p,t+i+1})+\log\beta-g(1-b_1^{es})+ \\ &+\frac{1}{2}var_{t+i}(r_{p,t++i1}-\gamma(c_{t+i+1}^{es}-x_{t+i}))-b_0^{es}+b_1^{es}(k^{es}-\rho_c^{es}b_0^{es})]. \end{aligned}$$

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