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Fluctuations in dilute antiferromagnets: Curie–Weiss models

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Abstract. We compute the fluctuations of the order parameter in the Curie–Weiss version of a site-dilute antiferromagnet. Our results show: (i) Gaussian fluctuations away from criticality or at a first-order critical point with sample and thermal fluctuations contributing in same order; (ii) Non-Gaussian fluctuations with critical exponents modified by the presence of dilution at the second-order critical point. In this case sample-induced fluctuations are enhanced to dominate over the thermal ones. Critical exponents are the same as in Curie–Weiss random field Ising model.

1. Introduction

Considerable theoretical effort has been made in recent years to understand the Ising model in the presence of a random magnetic field (RMF) [1–8]. However, random fields cannot be directly produced in laboratories. After the original paper by Fishman and Aharony [3] and the arguments of Wong *et al* [9], there is a generalized belief that this model is somehow equivalent to site-dilute antiferromagnetic Ising models in the presence of an applied uniform magnetic field (DAF), which are experimentally accessible systems [10]. Particularly, the degree of dilution and the intensity of the field, which are supposed to be related to the RMF parameters, can be well controlled.

With few exceptions [11] the works on this equivalence have been centred in the usual mean field approximation [3, 9, 12]. A complete mapping between the parameters and phase diagrams has been obtained [4] for Curie–Weiss (CW) versions of both models, which were solved [4, 6] by a method due to van Hemmen [13]. In spite of being mean field models, the latter are somewhat subtler from the probabilistic point of view. Rigorous work by Ellis and Newman [14–16] studying large deviation in classical Ising-like CW models has shown that they display non-trivial fluctuations of the order parameter at criticality. These results have been extended to disordered models such as RMF [1, 2].

In this work we study the fluctuations of the CW version of the DAF model and compare our results with those [1, 2] of the correspondent RMF model.

The Curie–Weiss DAF model we use is described in a finite volume $\Lambda \subset Z^d$ by the Hamiltonian

$$H_{\mathrm{DAF}} = -\frac{J_0}{2N}\sum_{i,j\in\Lambda_e}\xi_i\xi_j\sigma_i\sigma_j - \frac{J_0}{2N}\sum_{i,j\in\Lambda_0}\xi_i\xi_j\sigma_i\sigma_j + \frac{J}{N}\sum_{\substack{i\in\Lambda_e\\ j\in\Lambda_0}}\xi_i\xi_j\sigma_i\sigma_j + H\sum_{i\in\Lambda}\xi_i\sigma_i \tag{1}$$

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where $\Lambda_{e(o)} = \Lambda \cap Z^d_{e(o)}$ with Z^d_e (Z^d_o) being the sublattice of Z^d for which the sum of coordinates of each site are even (odd) integers. The interaction is antiferromagnetic ($J > 0$) between sites in different sublattices and there is an explicit ferromagnetic interaction ($J_0 \geq 0$) between sites in the same sublattice. The random variables $\xi_i \in \{0, 1\}$ describe the site dilution and they are taken to be independent and identically distributed, with

$$\xi_i = \begin{cases} 1 & \text{probability } p \\ 0 & \text{probability } 1-p. \end{cases}$$

The spin variables, σ_i, are, for simplicity, taken to be of Ising type: $\sigma_i = \pm 1$. The external magnetic field H is uniform and deterministic, and N denotes the number of points in Λ.

The Hamiltonian (1) is slightly different from that used in a previous work [4]; it permits, by making $J_0 = 0$, the study of a more natural situation where no explicit ferromagnetic interaction inside the sublattices is considered.

The RMF model to be compared with the model given by (1) is described by the Hamiltonian

$$H_{\mathrm{RMF}} = -\frac{J}{2N} \sum_{i,j \in \Lambda} \sigma_i \sigma_j + \sum_{i \in \Lambda} h_i \sigma_i \tag{2}$$

where h_i, $i \in \Lambda$, are independent identically distributed random variables, being equal to $\pm H$ with probability $\frac{1}{2}$.

This paper is organized as follows. In section 2 we compute the thermodynamics of the DAF model defined by (1) and compare this with the thermodynamics of the RMF defined by (2) as computed in [6] and [1]. In particular we recover their complete equivalence observed in [4] for $J_0 = J$. This thermodynamical equivalence however is somewhat misleading, as it remains true even if $p = 1$! The solution of this apparent paradox is presented in section 3 where we compute the asymptotics of the fluctuations of the order parameter for large N to verify the equivalence of both models for all values of J_0, $0 \leq J_0 \leq J$, only if $p \neq 1$. For $p = 1$, even if the two models are thermodynamically equivalent (for $J_0 = J$), the statistics of their fluctuation-variables and in particular their critical exponents are completely different. In particular for $0 < p < 1$ we obtain non-self-averaging (i.e. sample dependent) fluctuations. At the critical temperature, sample fluctuations dominate thermal fluctuations for large N, being of the same order for $T \neq T_c$. These are the results in [1].

2. Thermodynamics of the model

We compute, for both models, their free energy f given by

$$\beta f = \lim_{N \to \infty} -\frac{1}{N} \ln\left(\sum_{\{\sigma\}} e^{-\beta H} \right)$$

where β is the inverse of the temperature, $\{\sigma\}$ denotes all the possible spin configurations, and H is the Hamiltonian. Taking $H = H_{\mathrm{DAF}}$ as in (1), one may write

$$H_{\mathrm{DAF}} = \frac{J_1}{N} \left(\frac{S_e + S_o}{2} \right)^2 - \frac{J_2}{N} \left(\frac{S_e - S_o}{2} \right)^2 + H(S_e + S_o)$$

where

$$S_{e(o)}=\sum_{i\in\Lambda_{e(o)}}\xi_i\sigma_i \qquad \text{and} \qquad \begin{cases} J_1=J-J_0 \\ J_2=J+J_0. \end{cases}$$

Then

$$Z_{\mathrm{DAF}}^{(N)}=\sum_{\{\sigma\}}\mathrm{e}^{-\beta H_{\mathrm{DAF}}}=\frac{2^N N}{4\pi}\beta\sqrt{J_1J_2}\int \mathrm{d}m\,\mathrm{d}q\,\mathrm{e}^{-N\phi_{\mathrm{DAF}}^{(N)}(q,m)}$$

where

$$\phi_{\mathrm{DAF}}^{(N)}(q,m)=\frac{\beta}{2}\left(\frac{J_1q^2+J_2m^2}{2}\right)-\frac{F_{\mathrm{e}}^{(N)}}{2}\ln\cosh\left[\frac{\beta(J_2m-\mathrm{i}J_1q)-2\beta H}{2}\right]$$
$$-\frac{F_{\mathrm{o}}^{(N)}}{2}\ln\cosh\left[\frac{\beta(J_2m+\mathrm{i}J_1q)+2\beta H}{2}\right] \tag{3}$$

with

$$F_{\mathrm{e(o)}}^{(N)}=\frac{2}{N}\sum_{i\in\Lambda_{e(o)}}\xi_i.$$

Here we have twice made use of the identity

$$\exp(a^2)=\frac{1}{\sqrt{2\pi}}\int \mathrm{d}x\,\exp\left(-\frac{x^2}{2}+\sqrt{2}ax\right)$$

with

$$a=-\mathrm{i}\sqrt{\frac{\beta J_1}{N}}\left(\frac{S_{\mathrm{e}}+S_{\mathrm{o}}}{2}\right)$$

in one case and

$$a=\sqrt{\frac{\beta J_2}{N}}\left(\frac{S_{\mathrm{e}}-S_{\mathrm{o}}}{2}\right)$$

in the other, together with a suitable change of the integration variables.

It can be shown [17] that Laplace's asymptotic method is valid for multiple integrals, thus obtaining the following expression for free energy:

$$\beta f_{\mathrm{DAF}}(\beta,J,J_0,p,H)=\phi_{\mathrm{DAF}}(q^*,m^*)$$

where

$$\phi_{\mathrm{DAF}}(q,m)=\lim_{N\to\infty}\phi_{\mathrm{DAF}}^{(N)}(q,m)$$
$$=\frac{\beta}{2}\left(\frac{J_1q^2+J_2m^2}{2}\right)-\frac{p}{2}\left\{\ln\cosh\left[\frac{\beta(J_2m-\mathrm{i}J_1q)-2\beta H}{2}\right]\right.$$
$$\left.+\ln\cosh\left[\frac{\beta(J_2m+\mathrm{i}J_1q)+2\beta H}{2}\right]\right\} \tag{4}$$

and (q^*, m^*) is the saddle point of $\phi_{\mathrm{DAF}}(q,m)$.

In the new variables

$$m_\pm=\frac{m}{2}\pm\mathrm{i}\frac{q}{2}$$

the above expressions take the form

$$\beta f_{\mathrm{DAF}}(\beta, J, J_0, p, H)$$
$$= \frac{\beta J_0}{2}(m_+^2 + m_-^2) + \beta J m_+ m_- - \frac{p}{2}[\ln\cosh(\beta J_0 m_- + \beta J m_+ + \beta H)$$
$$+ \ln\cosh(\beta J_0 m_+ + \beta J m_- - \beta H)]$$

with $m_\pm$ defined by the equations

$$m_\pm = \frac{p}{2}\tanh(\beta J_0 m_\pm + \beta J m_\mp \mp \beta H).$$

This result can also be obtained with the use of van Hemmen's method as in [4]. In particular for $J_0 = J$ we have

$$\beta f_{\mathrm{DAF}}(\beta, J, J, p, H) = \tfrac{1}{2}\beta J M^2 - \frac{p}{2}[\ln\cosh(\beta J M + \beta H) + \ln\cosh(\beta J M - \beta H)] \tag{5}$$

with $M = m_+ + m_-$ defined by

$$M = \frac{p}{2}[\tanh(\beta J M + \beta H) + \tanh(\beta J M - \beta H)].$$

However, it is known [1, 6] that the free energy for the CW RMF model given by (2) is

$$\beta f_{\mathrm{RMF}}(\beta, J, H) = \tfrac{1}{2}\beta J M^2 - \tfrac{1}{2}[\ln\cosh(\beta J M + \beta H) + \ln\cosh(\beta J M - \beta H)] \tag{6}$$

with M determined by the equation

$$M = \tfrac{1}{2}[\tanh(\beta J M + \beta H) + \tanh(\beta J M - \beta H)].$$

From (5) and (6) it follows that

$$f_{\mathrm{DAF}}(\beta, J, J, p, H) = p f_{\mathrm{RMF}}(\beta, pJ, H) \tag{7}$$

for any $p \in (0, 1]$ (including the deterhinistic case $p = 1$!).

Remarks. (i) It may seem surprising that the equivalence holds true even for $p = 1$, the deterministic case. However we will show in section 3 that from the point of view of fluctuations the models with $p = 1$ and $0 < p < 1$ are drastically different, in particular with different critical exponents.

(ii) The exact mapping between the thermodynamics of the two models was only possible for $J_0 = J$. However we will show in section 3 that from the point of view of fluctuations the equality of critical exponents holds true even for $0 \leqslant J_0 < J$ $(p \neq 1)$.

The above remarks indicate that no great importance should be assigned to this thermodynamical equivalence.

3. Fluctuations

The study of fluctuations in the statistical mechanics of disordered systems is much more complicated than in non-random models. This remains true even for Curie-Weiss models. For the RMF model this has been rigorously discussed by Amaro de Matos and Perez [1] extending the techniques and ideas used by Ellis and Newman [14–16] in the study of non-random CW models.

Here we proceed to compute the asymptotics for large N, of the fluctuations of the order parameter in the DAF model. In [1] the reader will find the rigorous justifications for the heuristic consideration we will present here.

Let us first consider the case $J_0 = J$. We will later on show that regarding fluctuations, the models with $0 \leq J_0 \leq J$, are essentially equivalent.

The order parameter μ, the difference of magnetization in the two sublattices,

$$\mu_N = \frac{\Sigma_{\{\sigma\}}\{\exp(-\beta H_{\mathrm{DAF}})(S_e - S_o)/N\}}{\Sigma_{\{\sigma\}}\{\exp(-\beta H_{\mathrm{DAF}})\}}$$

in the limit $N \to \infty$ satisfies

$$\mu = \lim_{N \to \infty} \mu_N$$

where μ is defined by

$$\phi_{\mathrm{DAF}}(0, \mu) = \inf\{\phi_{\mathrm{DAF}}(0, m) \colon m \in \mathbb{R}\}.$$

The analysis of fluctuations of the order parameter consists then in the determination of the probability distribution in the limit $N \to \infty$ of the random variable:

$$y_N = N^{\gamma}\left(\frac{S_e - S_o}{N} - \mu\right) = \frac{(S_e - S_o) - N\mu}{N^{1-\gamma}}.$$

Here the value of γ is to be determined as to guarantee a non-trivial limit for the distribution of y_N.

The probability distribution of y_N for large N is related to the function

$$\phi^N_{\mathrm{DAF}}(0, m) = \beta J \frac{m^2}{2} - \frac{F_e^{(N)}}{2} \ln \cosh(\beta J m - \beta H) - \frac{F_o^{(N)}}{2} \ln \cosh(\beta J m + \beta H) \tag{8}$$

as follows [1]. Introducing an auxiliary Gaussian random variable W of zero mean and variance 1, i.e. $W \sim N(0, 1)$, independent of all other variables we have, for real a and γ:

$$\frac{W}{\sqrt{\beta J}\, N^{1/2-\gamma}} + \frac{(S_e - S_o) - Na}{N^{1-\gamma}} \sim \frac{\mathrm{d}x \exp\left\{-N\phi^{(N)}_{\mathrm{DAF}}\left(0, \dfrac{x}{N^{\gamma}} + a\right)\right\}}{\displaystyle\int \mathrm{d}x \exp\left\{-N\phi^{(N)}_{\mathrm{DAF}}\left(0, \dfrac{x}{N^{\gamma}} + a\right)\right\}} \tag{9}$$

where the RHS is the probability distribution of the random variable in the LHS. For a derivation of this formula see the appendix.

For large N, all relevant information is contained in what happens around the point μ_N, the minimum of $\phi^{(N)}_{\mathrm{DAF}}(0, m)$, i.e.

$$\phi^{(N)}_{\mathrm{DAF}}(0, \mu_N) = \inf\{\phi^{(N)}_{\mathrm{DAF}}(0, m) \colon m \in \mathbb{R}\}.$$

So we first compute fluctuations around μ_N, using (9) with $a = \mu_N$ and expanding

$$\phi^{(N)}_{\mathrm{DAF}}\left(0, \frac{x}{N^{\gamma}} + \mu_N\right)$$

around $x = 0$, so obtaining the asymptotic distribution of the random variable

$$z_N = \frac{(S_e - S_o) - N\mu_N}{N^{1-\gamma}}.$$

The random variable z_N will be said to represent the thermal fluctuations. Notice however that μ_N itself is a random variable because of the intrinsic randomness (dilution) of the function $\phi_{\mathrm{DAF}}^{(N)}(0, m)$ (see expression (3)), whose minimum is attained at μ_N. The fluctuations of μ_N around the asymptotic value μ (non-random!) will be called sample induced fluctuations. Therefore the y_N fluctuations will be obtained as a 'composition' of the z_N thermal fluctuations and the sample fluctuations of μ_N.

We begin with sample fluctuations from

$$\mu_N = \tfrac{1}{2}[\tanh(\beta J\mu_N - \beta H)F_{\mathrm{e}}^{(N)} + \tanh(\beta J\mu_N + \beta H)F_{\mathrm{o}}^{(N)}]$$

and

$$\mu = \frac{p}{2}[\tanh(\beta J\mu - \beta H) + \tanh(\beta J\mu + \beta H)].$$

First, the law of large numbers guarantees that $\mu_N \to \mu$ with probability one. Expanding then $\tanh(\beta J\mu_N \pm \beta H)$ around μ we obtain for $T \geq T_c$, where $\mu = 0$, the following expression:

$$\begin{aligned}\frac{\phi_{\mathrm{DAF},2}(0,0)}{\beta J}\mu_N &= \tfrac{1}{2}\tanh(\beta H)(F_{\mathrm{o}}^{(N)} - F_{\mathrm{e}}^{(N)}) + \frac{\beta J}{2}\operatorname{sech}^2(\beta H)(F_{\mathrm{o}}^{(N)} + F_{\mathrm{e}}^{(N)} - 2p)\mu_N \\ &\quad - \frac{(\beta J)^2}{2}\operatorname{sech}^2(\beta H)\tanh(\beta H)(F_{\mathrm{o}}^{(N)} - F_{\mathrm{e}}^{(N)})\mu_N^2 - \frac{\phi_{\mathrm{DAF},4}(0,0)}{3!\beta J}\mu_N^3 \\ &\quad - \frac{\phi_{\mathrm{DAF},4}(0,0)}{3!2p\beta J}(F_{\mathrm{o}}^{(N)} + F_{\mathrm{e}}^{(N)} - 2p)\mu_N^3 + \ldots \end{aligned} \tag{10}$$

where $\phi_{\mathrm{DAF},j}$ is the derivative of j-order of ϕ_{DAF}.

Now, since $F_{\mathrm{e(o)}}^{(N)}$ are sums of independent identically distributed random variables converging both to p (the dilution), we obtain from the central limit theorem:

$$\tfrac{1}{2}\tanh(\beta H)(F_{\mathrm{o}}^{(N)} - F_{\mathrm{e}}^{(N)}) \underset{N\to\infty}{\simeq} \frac{U_1}{\sqrt{N}} \tag{11}$$

where

$$U_1 \sim N(0, \sigma_1^2)$$

$$\sigma_1^2 = \frac{p(1-p)}{2}\tanh^2(\beta H).$$

We now define the 'type' of the minimum μ of $\phi_{\mathrm{DAF}}(0, m)$, as the smallest integer k such that $\phi_{\mathrm{DAF},2k}(0, \mu) \neq 0$. From (10) and (11) it then follows that for $T \geq T_c$, i.e. $\mu = 0$, the sample fluctuations are given by:

(i) for $k = 1$, i.e. away from criticality or at a first-order critical point

$$\mu_N \underset{N\to\infty}{\simeq} \frac{\beta J}{\phi_{\mathrm{DAF},2}(0,0)}\frac{U_1}{\sqrt{N}} \tag{12a}$$

(ii) for $k = 2$, i.e. at a second-order critical point

$$\mu_N = \left[\frac{3!\beta J}{\phi_{\mathrm{DAF},4}(0,0)}\frac{U_1}{\sqrt{N}}\right]^{1/3} \tag{12b}$$

i.e.

$$N^{\rho(k)}\mu_N \underset{N\to\infty}{\simeq} V_k \tag{13}$$

where

$$\rho(k)=\frac{1}{2(2k-1)}$$

and

$$V_k=\begin{cases}\dfrac{\beta J}{\phi_{\mathrm{DAF},2}(0,0)}\,U_1 & \text{for } k=1\\ \left[\dfrac{3!\beta J}{\phi_{\mathrm{DAF},4}(0,0)}\,U_1\right]^{1/3} & \text{for } k=2.\end{cases}$$

Let us now deal with thermal fluctuations. From (9) with $a=\mu_N$ they are given by:

$$\frac{W}{\sqrt{\beta J}\,N^{1/2-\gamma}}+\frac{(S_e-S_o)-N\mu_N}{N^{1-\gamma}}\sim\frac{dx\,\exp\left\{-N\phi^{(N)}_{\mathrm{DAF}}\left(0,\dfrac{x}{N^\gamma}+\mu_N\right)\right\}}{\int dx\,\exp\left\{-N\phi^{(N)}_{\mathrm{DAF}}\left(0,\dfrac{x}{N^\gamma}+\mu_N\right)\right\}}. \tag{14}$$

We then expand

$$\phi^{(N)}_{\mathrm{DAF}}\left(0,\frac{x}{N^\gamma}+\mu_N\right)$$

around $x=0$ to obtain

$$\phi^{(N)}_{\mathrm{DAF}}\left(0,\frac{x}{N^\gamma}+\mu_N\right)=\phi^{(N)}_{\mathrm{DAF}}(0,\mu_N)+\frac{1}{2N^{2\gamma}}\,\phi^{(N)}_{\mathrm{DAF},2}(0,\mu_N)x^2+\dots \tag{15}$$

Notice that $\phi^{(N)}_{\mathrm{DAF},1}(0,\mu_N)=0$ since μ_N is a point of minimum for $\phi^{(N)}_{\mathrm{DAF}}(0,m)$. Then we expand $\phi^{(N)}_{\mathrm{DAF},j}(0,\mu_N)$ as a power series in μ_N (i.e. around $\mu=0$). For instance,

$$\phi^{(N)}_{\mathrm{DAF},2}(0,\mu_N)=\phi^{(N)}_{\mathrm{DAF},2}(0,0)+\phi^{(N)}_{\mathrm{DAF},3}(0,0)\mu_N+\tfrac{1}{2}\phi^{(N)}_{\mathrm{DAF},4}(0,0)\mu_N^{\,2}+\dots$$

Now it is crucial to notice that $\phi^{(N)}_{\mathrm{DAF},j}(0,0)$ is a sum of independent identically distributed random variables, and so using the central limit theorem we have:

$$\phi^{(N)}_{\mathrm{DAF},j}(0,0)\underset{N\to\infty}{\simeq}\phi_{\mathrm{DAF},j}(0,0)+\beta J\frac{U_j}{\sqrt{N}}$$

where $U_j\sim N(0,\sigma_j^2)$. Therefore we obtain:

(i) for $k=1$ (i.e. $\phi_{\mathrm{DAF},2}(0,0)>0$)

$$\phi^{(N)}_{\mathrm{DAF},2}(0,\mu_N)\underset{N\to\infty}{\simeq}\phi_{\mathrm{DAF},2}(0,0)+\beta J\frac{U_2}{\sqrt{N}} \tag{16}$$

(ii) for $k=2$ (i.e. $\phi_{\mathrm{DAF},2}(0,0)=\phi_{\mathrm{DAF},3}(0,0)=0$ and $\phi_{\mathrm{DAF},4}(0,0)>0$)

$$\phi^{(N)}_{\mathrm{DAF},2}(0,\mu_N)\underset{N\to\infty}{\simeq}\beta J\frac{U_2}{\sqrt{N}}+\frac{\phi_{\mathrm{DAF},4}(0,0)}{2}\left[\frac{3!\beta J}{\phi_{\mathrm{DAF},4}(0,0)}\frac{U_1}{\sqrt{N}}\right]^{2/3}. \tag{17}$$

We then go back to (14) using (15), (16) or (17) to see that:

(i) for $k=1$ ($\gamma=\frac{1}{2}$)

$$\lim_{N\to\infty} z_N = \lim_{N\to\infty} \frac{(S_e-S_o)-N\mu_N}{N^{1-\gamma}} \sim \exp\left\{-\left[\frac{1}{\phi_{\mathrm{DAF},2}(0,)}-1\right]^{-1}\frac{x^2}{2}\right\}\mathrm{d}x \tag{18a}$$

(ii) for $k=2$ ($\gamma=\frac{1}{3}$)

$$\lim_{N\to\infty} z_N \sim \exp\left\{-\frac{\phi_{\mathrm{DAF},4}(0,0)}{2}\left[\frac{3!\beta J}{\phi_{\mathrm{DAF},4}(0,0)}U_1\right]^{2/3}\frac{x^2}{2}\right\}\mathrm{d}x \tag{18b}$$

i.e.

$$z_N = N^{\gamma(k)}\left(\frac{S_e-S_o}{N}-\mu_N\right) \underset{N\to\infty}{\cong} T_k \tag{19}$$

where

$$\gamma(k)=\begin{cases}\frac{1}{2} & \text{for } k=1\\ \frac{1}{3} & \text{for } k=2\end{cases}$$

and T_k is a Gaussian of zero mean.

Comparing (13) and (19) we see that the sample and thermal fluctuations contribute in the same order for $k=1$ (i.e. away from criticality or at a first-order critical point), $\gamma(1)=\rho(1)=\frac{1}{2}$, with Gaussian distributions. For $k=2$ (i.e. at a second-order critical point) however, the sample fluctuations dominate over the thermal ones: $\gamma(2)=\frac{1}{3}$, $\rho(2)=\frac{1}{6}$. In conclusion

$$\frac{S_e-S_o}{N} \underset{N\to\infty}{\simeq} \frac{V_k}{N^{\rho(k)}}+\frac{T_k}{N^{\gamma(k)}}$$

therefore

$$\lim_{N\to\infty} y_N = \begin{cases} U \sim N\left(\dfrac{\beta J U_1}{\phi_{\mathrm{DAF},2}(0,0)}, \dfrac{1}{\phi_{\mathrm{DAF},2}(0,0)}-1\right) & \text{for } k=1\\ V_2 & \text{for } k=2.\end{cases}$$

Remarks. (i) Although for $k=2$ we are not in a central limit situation, the asymptotic distribution of the sample fluctuations are Gaussian, with non-Gaussian critical exponent!

(ii) The above results show in particular that the fluctuations of the order parameter are sample dependent in all cases. For $k=1$ the thermal fluctuations contribute in the same order whereas for $k=2$ the sample induced fluctuations, due to the dilution, are enhanced and dominate over the thermal fluctuation.

(iii) Fluctuations of the order parameter in the RMF defined by (2) have been computed with the same methods by Amaro de Matos and Perez [1, 2]. They obtain the same critical exponents and probability distributions of the same nature as the ones above, for both $k=1$ and $k=2$.

Finally we discuss the case $0 \leq J_0 < J$. In this case we consider the saddle points of $\phi^{(N)}_{\mathrm{DAF}}(q, m)$ (equation (3)) and $\phi_{\mathrm{DAF}}(q, m)$ (equation (4)); they are (q^*_N, m^*_N) and

(q^*, m^*) given respectively by:

$$q_N^* = \mathrm{i}\left[\frac{F_o^{(N)}}{2}\tanh\left(\frac{\beta J_2 m_N^* + \mathrm{i}\beta J_1 q_N^* + 2\beta H}{2}\right) - \frac{F_e^{(N)}}{2}\tanh\left(\frac{\beta J_2 m_N^* - \mathrm{i}\beta J_1 q_N^* - 2\beta H}{2}\right)\right]$$

$$m_N^* = \frac{F_o^{(N)}}{2}\tanh\left(\frac{\beta J_2 m_N^* + \mathrm{i}\beta J_1 q_N^* + 2\beta H}{2}\right) + \frac{F_e^{(N)}}{2}\tanh\left(\frac{\beta J_2 m_N^* - \mathrm{i}\beta J_1 q_N^* - 2\beta H}{2}\right)$$

and

$$q^* = \mathrm{i}\frac{p}{2}\left[\tanh\left(\frac{\beta J_2 m^* + \mathrm{i}\beta J_1 q^* + 2\beta H}{2}\right) - \tanh\left(\frac{\beta J_2 m^* - \mathrm{i}\beta J_1 q^* - 2\beta H}{2}\right)\right]$$

$$m^* = \frac{p}{2}\left[\tanh\left(\frac{\beta J_2 m^* + \mathrm{i}\beta J_1 q^* + 2\beta H}{2}\right) + \tanh\left(\frac{\beta J_2 m^* - \mathrm{i}\beta J_1 q^* - 2\beta H}{2}\right)\right].$$

The law of large numbers guarantees again that in the limit $N \to \infty$, $m_N^* \to m^*$ and $q_N^* \to q^*$ with probability one. The fact that $Z_{\mathrm{DAF}}^{(N)}$ is real implies the existence of a unique q^* given by: $q^* = \mathrm{i}q_0$ and $q_0 < 0$. Thus, expanding, as before,

$$\tanh\left(\frac{\beta J_2 m_N^* \pm \mathrm{i}\beta J_1 q_N^* \pm 2\beta H}{2}\right)$$

around (q^*, m^*) for $\beta \leqslant \beta_c$ (i.e. $m^* = 0$), we obtain:

$$\begin{aligned}
&\left(\frac{\partial^2 \phi_{\mathrm{DAF}}}{\partial q^2}\right)_* (q_N^* - q^*) \\
&\quad = \mathrm{i}A_1(E_o^{(N)} + E_e^{(N)})T(+) + (\mathrm{i}A_1)^2(E_o^{(N)} + E_e^{(N)})T'(+)(q_N^* - q^*) \\
&\qquad + \mathrm{i}A_1A_2(E_o^{(N)} - E_e^{(N)})T'(+)m_N^* - \left(\frac{\partial^3 \phi_{\mathrm{DAF}}}{\partial q^3}\right)_* \frac{(q_N^* - q^*)^2}{2!} \\
&\qquad + (\mathrm{i}A_1)^3(E_o^{(N)} + E_e^{(N)})T''(+)\frac{(q_N^* - q^*)^2}{2!} - \left[\frac{\partial}{\partial q}\left(\frac{\partial^2 \phi_{\mathrm{DAF}}}{\partial m^2}\right)\right]_* \frac{(m_N^*)^2}{2!} \\
&\qquad + \mathrm{i}A_1A_2^2(E_o^{(N)} + E_e^{(N)})T''(+)\frac{(m_N^*)^2}{2!} \\
&\qquad + (\mathrm{i}A_1)^2A_2(E_o^{(N)} - E_e^{(N)})T''(+)(q_N^* - q^*)m_N^* \\
&\qquad - \left(\frac{\partial^4 \phi_{\mathrm{DAF}}}{\partial q^4}\right)_* \frac{(q_N^* - q^*)^3}{3!} + (\mathrm{i}A_1)^4(E_o^{(N)} + E_e^{(N)})\frac{(q_N^* - q^*)^3}{3!}T'''(+) \\
&\qquad + \mathrm{i}A_1(A_2)^3(E_o^{(N)} - E_e^{(N)})T'''(+)\frac{(m_N^*)^3}{3!} \\
&\qquad + (\mathrm{i}A_1)^3A_2(E_o^{(N)} - E_e^{(N)})T'''(+)\frac{(q_N^* - q^*)^2 m_N^*}{2} \\
&\qquad - \left[\frac{\partial^2}{\partial m^2}\left(\frac{\partial^2 \phi_{\mathrm{DAF}}}{\partial q^2}\right)\right]_* \frac{(m_N^*)^2(q_N^* - q^*)}{2} \\
&\qquad + (\mathrm{i}A_1)^2(A_2)^2(E_o^{(N)} + E_e^{(N)})T'''(+)\frac{(m_N^*)^2(q_N^* - q^*)}{2} + \ldots
\end{aligned}$$

and

$$\left(\frac{\partial^2\phi_{\mathrm{DAF}}}{\partial m^2}\right)_* m_N^*$$
$$= A_2(E_o^{(N)} - E_e^{(N)})T(+) + (A_2)^2(E_o^{(N)} + E_e^{(N)})T'(+)m_N^*$$
$$+ \mathrm{i}A_1A_2(E_o^{(N)} - E_e^{(N)})T'(+)(q_N^* - q^*) + (A_2)^3(E_o^{(N)} - E_e^{(N)})T''(+)(m_N^*)^2$$
$$+ (\mathrm{i}A_1)^2A_2(E_o^{(N)} - E_e^{(N)})T''(+)\frac{(q_N^* - q^*)^2}{2}$$
$$-\left[\frac{\partial}{\partial q}\left(\frac{\partial^2\phi_{\mathrm{DAF}}}{\partial m^2}\right)\right]_*(q_N^* - q^*)m_N^*$$
$$+ \mathrm{i}A_1(A_2)^2(E_o^{(N)} + E_e^{(N)})T''(+)(q_N^* - q^*)m_N^* - \left(\frac{\partial^4\phi_{\mathrm{DAF}}}{\partial m^4}\right)_*\frac{(m_N^*)^3}{3!}$$
$$+ (A_2)^4(E_o^{(N)} + E_e^{(N)})T'''(+)\frac{(m_N^*)^3}{3!}$$
$$+ (\mathrm{i}A_1)^3A_2(E_o^{(N)} - E_e^{(N)})T'''(+)\frac{(q_N^* - q^*)^3}{3!}$$
$$-\left[\frac{\partial^2}{\partial q^2}\left(\frac{\partial^2\phi_{\mathrm{DAF}}}{\partial m^2}\right)\right]_*\frac{(q_N^* - q^*)^2 m_N^*}{2}$$
$$+ (\mathrm{i}A_1)^2(A_2)^2(E_o^{(N)} + E_e^{(N)})T'''(+)\frac{(q_N^* - q^*)^2 m_N^*}{2}$$
$$+ (\mathrm{i}A_1)(A_2)^3(E_o^{(N)} - E_e^{(N)})T'''(+)\frac{(m_N^*)^2(q_N^* - q^*)}{2} + \ldots$$

where

$$A_{1(2)} = \frac{\beta J_{1(2)}}{2}$$
$$E_{\mathrm{e(o)}}^{(N)} = \frac{F_{\mathrm{e(o)}}^{(N)} - p}{2}$$

and

$$T(+) = \tanh(\mathrm{i}A_1 q^* + \beta H).$$

Since

$$\left(\frac{\partial^2\phi_{\mathrm{DAF}}}{\partial q^2}\right)_* = A_I[1 + A_I p\,\mathrm{sech}^2(\mathrm{i}A_I q^* + \beta H)] > 0$$

there is no criticality associated with the parameter q.

This implies the behaviour

$$q_N^* - q^* \underset{N\to\infty}{\simeq} \frac{\text{Gaussian}}{\sqrt{N}}.$$

Away from criticality (for m) we have $(\partial^2\phi_{\mathrm{DAF}}/\partial m^2)_* \neq 0$, and from the expansion for m_N^*, results in

$$m_n^* \underset{N\to\infty}{\simeq} \frac{\text{Gaussian}}{\sqrt{N}}.$$

The above shows therefore that the rate of approach of m_N^* to m^* as $N \to \infty$ is the same as in the case $J_0 = J$, in particular we get the same critical exponents and asymptotic probability distributions at both $k = 1$ and $k = 2$.

Appendix. The probability distribution of the order parameter

The derivation of formula (9) we present here is adapted from [1, 14–16]. Let us compute the characteristic function $K_n(t)$ of the random variable $A_N = [(S_e - S_o) - Na]/N^{1-\gamma}$:

$$K_n(t) = \frac{1}{Z^{(N)}} \sum_{\{\sigma\}} \exp(\mathrm{i}\, A_N t)\, \mathrm{e}^{-\beta H(\sigma)} \tag{A1}$$

where $H = H_{\mathrm{DAF}}$ with $J_1 = 0$, $J_2 = 2J$ (i.e. $J_0 = J$). In this case we compute the numerator of (A1):

$$\begin{aligned}
&\sum_{\{\sigma\}} \exp(\mathrm{i}\, A_N t)\, \mathrm{e}^{-\beta H(\sigma)} \\
&\quad = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\, \mathrm{e}^{-x^2/2} \sum_{\{\sigma\}} \left\{ \exp\left[\left(\sqrt{\frac{\beta J}{N}}\, x - \beta H + \frac{\mathrm{i}t}{N^{1-\gamma}}\right) S_e\right]\right\} \\
&\qquad \times \left\{ \exp\left[-\left(\sqrt{\frac{\beta J}{N}}\, x + \beta H + \frac{\mathrm{i}t}{N^{1-\gamma}}\right) S_o\right]\right\} \\
&\quad = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\, \mathrm{e}^{-x^2/2} \exp\left[\frac{N}{2} F_e^{(N)} \ln\cosh\left(\sqrt{\frac{\beta J}{N}}\, x - \beta H + \frac{\mathrm{i}t}{N^{1-\gamma}}\right)\right] \\
&\qquad \times \exp\left[\frac{N}{2} F_o^{(N)} \ln\cosh\left(\sqrt{\frac{\beta J}{N}}\, x + \beta H + \frac{\mathrm{i}t}{N^{1-\gamma}}\right)\right] \\
&\quad = \left(\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\beta J N}}{N^{\gamma}} \int \mathrm{d}x\, \exp\left[-N\phi^{(N)}\left(\frac{x}{N^{\gamma}} + a\right)\right]\right) \exp\left(\frac{t^2}{2\beta J N^{1-2\gamma}}\right)
\end{aligned} \tag{A2}$$

where

$$\phi^{(N)}(x) = \phi^{(N)}_{\mathrm{DAF}}(0, x) = \frac{\beta J x^2}{2} - \frac{F_e^{(N)}}{2} \ln\cosh\beta(Jx - H) - \frac{F_o^{(N)}}{2} \ln\cosh\beta(Jx + H).$$

The denominator of (A1) is obtained by setting $t = 0$ in (A2) and therefore

$$K_n(t) \exp\left(-\frac{t^2}{2\beta J N^{1-2\gamma}}\right) = \frac{\int \mathrm{e}^{\mathrm{i}tx} \exp\left[-N\phi^{(N)}\left(\frac{x}{N^{\gamma}} + a\right)\right] \mathrm{d}x}{\int \exp\left[-N\phi^{(N)}\left(\frac{x}{N^{\gamma}} + a\right)\right] \mathrm{d}x}. \tag{A3}$$

Let now W be a Gaussian random variable of zero mean and variance 1 independent of all other variables involved in the problem. The characteristic function of $W/\sqrt{\beta J} N^{1/2-\gamma}$ is $\exp(-t^2/2\beta N^{1-2\gamma})$. Now the characteristic function of a sum of independent random variables is the product of their characteristic functions and so the left-hand side of (A3) is the characteristic function of

$$\frac{W}{\sqrt{\beta J}\,N^{1/2-\gamma}}+\frac{(S_e-S_o)-Na}{N^{1-\gamma}}$$

thus proving (9).

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