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Consuming durable goods when stock markets jump: A strategic asset allocation approach

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ABSTRACT

In this paper we show the impact of considering jumps in the return process of risky assets when deciding how to invest and consume throughout time. Agents derive their utilities from consumption over time. We consider an agent that invests in the financial market and in durable and perishable consumption goods. Assuming that there are costs for transacting the durable good, we show that an agent who does not consider the possibility of jumps will make suboptimal decisions, not only regarding the fraction of wealth invested in the stock market, but also with respect to the timing for trading on the durable good. Furthermore we also show that jumps cause a non-obvious asymmetric impact on the thresholds that lead the consumer to trade the durable good, even when the jump distribution is symmetric.

1. Introduction

In times of financial crisis such as the one that we are living, the role of downward jumps in financial decisions becomes particularly relevant. A quick look at the performance of financial markets in the recent past makes this point clear, when the intensity of jumps has been particularly high.

In this paper we aim to analyze the impact of such jumps in the investment and consumption decisions, particularly when agents consume durable goods for which there are transaction costs.

The main problem faced by investors is the uncertainty regarding their future income and capacity to consume. Such uncertainty is typically characterized by the first two moments of the returns distribution. However, in the presence of jumps, higher moments are affected and the returns distribution becomes skewed and leptokurtic, strongly affecting the investment decisions.

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As an example of a common durable good we can think about the investing problem of the owner of a house. The decision to sell the house happens when the ratio wealth to house value is below a certain threshold. When there is the possibility of significant jumps in the stock market, the higher moments of the stock return distribution change, which leads to changes, not only in the optimal perishable good consumption and stock market investment strategy, but also in the optimal durable good trading strategy. We show that the boundaries at which the individual chooses to trade the durable good change in a non-obvious asymmetric way, due to the combination of the direct impact of jumps on the statistical properties of stock returns and the indirect effect of jumps on the optimal stock investment strategy.

We followed Damgaard et al. (2003) who considered a similar problem in a market where risky assets prices evolve according to a geometric Brownian motion. Given the extensive evidence of non-normality in stock market returns in the financial literature, we consider an extension of this model that includes jumps in stock market prices.

Stock market returns distributions are usually left skewed and leptokurtic,[1] which suggests the existence of jumps. There is a wide array of papers in the financial literature that empirically confirm the existence of jumps in stock market returns such as Andersen et al. (2002), Eraker et al. (2003) and Jarrow and Rosenfeld (1984) who analyze daily time-series of several American stock market indexes and find evidence of jumps in stock market returns. Also, Dunham and Friesen (2007) and Lee and Mykland (2008) among others used ultra-high frequency data on the S&P 500 and concluded that there were jumps in the index returns. Ait-Sahalia et al. (2001), Carr and Wu (2003), Jackwerth and Rubinstein (1996) and Pan (2002) studied the jump-risk premia implicit in the S&P 500 options and also found evidence of jumps on the underlying index distribution. Bollerslev and Todorov (2011), Kelly (2011) and Zhang et al. (2009), using different methods, have shown that jumps play a crucial role in explaining the equity risk premium, and also the price of several securities, such as index options and credit default swaps.

To our knowledge, not many papers have focused on optimal portfolio selection with transaction costs in a stock market with jumps.[2] Our paper is the first to analyze this problem, where jumps are driven by a Lévy process, and in the presence of both perishable and durable consumption goods. In order to understand the contribution of this paper, we briefly describe the evolution in the literature.

The problem of finding closed-formed solutions for the optimal investment strategy, in a model where stock markets jump, is not an easy one. Ait-Sahalia et al. (2009) accomplished this task, for specific types of utility functions and jump processes, and found that jumps decrease the investor's exposure to risky assets. The impact of jumps on stock markets investment was also studied by Liu et al. (2003), who conclude that jumps can lead to a substantial decrease in stock market investment, particularly for individuals with low risk-aversion. The equivalent wealth losses from ignoring jumps were estimated in Ascheberg et al. (2013) and in Das and Uppal (2004). These authors conclude that these losses are generally small, but they can be substantial for individuals with a relatively low risk-aversion.

Merton (1969) studied the optimal investment and consumption problem of an individual who consumes only a perishable good with no transaction costs. He assumed that the agent could invest in a riskless asset and a risky asset, whose price process follows a (continuous) geometric Brownian motion. Ignoring transaction costs and other market imperfections, he concluded that a CRRA consumer should invest a constant fraction of his wealth in the risky asset. Obviously, this strategy is not optimal for an investor who faces transaction costs whenever he trades the risky stock since such a strategy would involve continuous trading. Since Merton (1969), a vast number of papers focused in transaction costs and/or other market imperfections. Among others Davis and Norman (1990) and Shreve and Soner (1994) studied the problem of an infinitely lived investor facing proportional transaction costs when trading the only risky asset available in the economy. They showed that there is a wedge shaped region where it is optimal for the investor not to trade the risky asset. Whenever the risky asset investment becomes sufficiently low (high) relative to the riskless investment, the investor buys (sells) the risky asset in order to return to the no-trading region limit. Akian et al. (1996) extended the previous works by considering that the individual can invest in n risky assets concluding that, almost surely, the investor never trades more than one risky asset simultaneously. Liu (2004) considered n risky assets, but with fixed transaction costs and shows that, if risky assets' returns are uncorrelated, the optimal investment policy is to keep the amount invested in each risky asset between two constant levels. Whenever either of these bounds is reached, the agent trades to the corresponding optimal targets. Chellathurai and Draviam (2005, 2007), Dai and Yi (2009), Liu and Loewenstein (2002) and Zakamouline (2002, 2005) considered the optimal investment problem of a finite horizon individual, under several specifications for the transaction costs, and concluded that the no-trading region widens as the horizon gets shorter.

In our paper we use very similar techniques to measure the impact of jumps in the financial markets together with the simultaneous consumption of durable and perishable goods. The importance of considering both goods at the same time is that it allows for analyzing the impact of different transaction cost structures in the presence of the financial market prices' discontinuities. As we show in the remaining of the paper, the combination of these effects implies quite subtle changes in the investment strategy that has been over-regarded in the literature. Not only the presence of jumps affects the investment strategy in the stock market, as one would expect, but also affects the shape of the no-trading region for the durable consumption good. The presence of jumps has two different effects. A first direct effect is that jumps may force the investor to trade in a very unfavorable position, too distant from the boundaries of the no-trading region – leading to narrower

[1] See, for example, Andersen et al. (2002).

[2] Benth et al. (2002) and Framstad et al. (1999) are two exceptions.

boundaries. A second effect is that, as referred in Liu et al. (2003), the distribution of returns on the stock market becomes riskier, reducing the fraction of wealth invested in the capital markets. The impact of this effect is two-fold: on the one hand it reduces the volatility of the wealth to durable good ratio, reinforcing the first direct effect above; on the other hand, it reduces the growth rate of wealth, thus making more likely the wealth process to reach the lower boundary than the upper boundary – inducing as a final result a shift to the left of the no-trading region. The shape of this region can thus be shown to be jointly driven by the jump modelling and the transaction cost structure.

This paper is organized as follows. The next section presents the basic model; Section 3 solves the model in the presence of jumps but in the absence of transaction costs; Section 4 presents the solution to the problem with transaction costs; Section 5 characterizes the optimal investment and consumption strategies when transaction costs and jumps coexist; Section 6 presents a simulation with parameters that are standard in the literature in order to understand the magnitude and the impact of the effects described in this paper; Section 7 develops a sensitivity analysis of the parameters of our numerical experiment; Section 8 finally concludes the paper.

2. The model

We consider an economy with two kinds of consumption goods, a perishable good and a durable one, and with two financial assets, a riskless bond B and a risky stock S. The price processes are defined taking the perishable good price as the numeraire. The bond B pays a constant, continuously compound interest rate r. The stock S pays no dividend, has a *cadlag* price process that follows a geometric Lévy process

$$dS_t = \mu S_t\, dt + \sigma S_t\, dw_{1t} + S_{t^-} \int_{-1}^{L} \eta \tilde{N}(dt, d\eta)$$

where $\mu > r$ and $\sigma > 0$, $L > 0$ are constants, w_{1t} is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and

$$\tilde{N}(t, A) = N(t, A) - tq(A); \quad t \geq 0, \quad A \in B(-1, L)$$

is the compensator of a homogenous Poisson random measure $N(t, A)$ on $\mathbb{R}^+ \times B(-1, L)$ with intensity measure $E[N(t, A)] = tq(A)$, where q is the Lévy measure associated to N and $B(-1, L)$ denotes the Borel σ-algebra on $(-1, L)$. We assume that

$$\| q \| \equiv q((-1, L)) < \infty$$

Note that, since we assume that jump sizes are always greater than -1, the risky asset price process remains positive for all $t \geq 0$ *a.s.*

We assume, as in Damgaard et al. (2003), that the unit price of the durable good, P_t, follows a geometric Brownian motion

$$dP_t = P_t[\mu_P\, dt + \sigma_{P_1}\, dw_{1t} + \sigma_{P_2}\, dw_{2t}]$$

where $\mu_P > 0$, $\sigma_{P_1} > 0$ and $\sigma_{P_2} > 0$ are constant scalars, and w_{2t} is a Wiener process uncorrelated with w_{1t}. Note that it is impossible to hedge perfectly the risk associated with the durable price process by trading the risky asset, since their price evolution is only partially correlated.

We also assume that the stock of the durable good depreciates at a constant rate δ. This means that at any given time t the stock of the durable good evolves according to the following equation:

$$dK_t = -\delta K_t\, dt$$

if that good is not traded.

Finally, as Damgaard et al. (2003) and Grossman and Laroque (1990), we also assume that trading the durable good is costly. More precisely, each time the consumer trades the durable good he must pay a fee proportional to its pre-existing stock. Cuoco and Liu (2000) assume that the transaction cost is proportional to the change in the durable good stock. The first specification is more appropriate for durable goods such as a house or a car, since when a consumer chooses to change the stock of one such durable good, he usually sells it to buy a new one. The later assumption is more appropriate for goods such as furniture, whose stock is typically increased without selling any pre-existing furniture.

The agent must choose a consumption pattern for the perishable good, and a trading strategy for the durable good and the financial assets. Denoting by C_t his consumption rate of the perishable consumption good at time t, we assume that it is a progressively measurable process $C \in \mathcal{L}^1$ where

$$\mathcal{L}^q = \left\{ \{\mathcal{F}_t\} - \text{adapted processes } X\text{: } \int_0^T \| X_u(\omega) \|\, du < \infty \text{ for } P-a.e.\ \omega \in \Omega,\ T > 0 \right\}$$

We represent by Θ_{0t} and Θ_{1t} the amount invested in the riskless asset and the risky asset at time t, respectively. We rule out the possibility of short-selling the risky asset. Therefore, the set of trading strategies consists of the 3-dimensional progressively measurable stochastic processes (Θ_0, Θ_1, K) valued in $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ such that $\Theta_0 \in \mathcal{L}^1$, $\Theta_1 \in \mathcal{L}^2$ and $K \in \mathcal{L}^2$.

We define the agent's wealth X_t as the sum of his investments in the financial assets plus his durable good investments

$$X_t = \Theta_{0t} + \Theta_{1t} + K_t P_t$$

If he follows a trading strategy (Θ_0, Θ_1, K), then his wealth evolves according to

$$dX_t = [r(X_t - K_t P_t) + \Theta_{1t}(\mu - r) + (\mu_P - \delta)K_t P_t - C_t]\,dt$$
$$+ (\Theta_{1t}\sigma + K_t P_t \sigma_{P1})\,dw_{1t} + K_t P_t\,dw_{2t} + \Theta_t^- \int_{-1}^{L} \eta \tilde{N}(dt, d\eta)$$

whenever there is no durable good trade, and

$$X_\tau = X_{\tau^-} - \lambda K_{\tau^-} P_\tau$$

for every τ where the consumer changes his durable good stock, where λ represents the proportional cost of trading the durable good.

We require that the strategies followed by the investor do not lead to bankruptcy. In other words, his wealth must always be higher than the transaction cost he would face if he immediately sold the durable good. Therefore, he must follow a trading strategy such that, for every $t \geq 0$, the set (X_t, K_t, P_t) must belong *a.s.* to the solvency region

$$S = \{(x, k, p) \in \mathbb{R}_+^3 : x > \lambda kp\}.$$

Let (x, k, p) denote the initial values (X_0, K_0, P_0). It is natural to assume that if $(x, k, p) \in S$, then *a.s.* for every t, $(X_t, K_t, P_t) \in S$. Knowing that the risky asset price process may jump, we must strengthen the previous condition by requiring that the trading and perishable consumption strategies (Θ_1, K, C) are such that $(X_{t^-} - \Theta_1 \eta, K_t, P_t)$ belongs to S, for every $t \geq 0$, and $\eta \in (-1, L)$. We denote the set of admissible strategies (those that satisfy the previous conditions) by $A(x, k, p)$. We will assume that this set is non-empty.

We consider a consumer who maximizes an infinite horizon time-separable utility function that depends both on his perishable consumption flow and durable good stock

$$\int_0^\infty e^{-\rho t} U(C_t, K_t)\,dt$$

where $\rho > 0$ is a constant scalar representing the consumer's time preference. We assume that his instantaneous utility function is of the multiplicatively separable isoelastic form, as in Damgaard et al. (2003) and Kraft and Munk (2011),

$$U(c, k) = \frac{(c^\beta k^{1-\beta})^{1-\gamma}}{1-\gamma}$$

where β and $\gamma \in (0, 1)$. If the consumer follows the admissible trading and perishable consumption strategy (Θ_1, K, C) his intertemporal expected utility will be

$$J^{\Theta_1, K, C}(x, k, p) = E\left[\int_0^\infty e^{-\rho t} U(C_t, K_t)\,dt\right]$$

The objective of this agent is to choose an admissible trading and perishable consumption strategy $(\Theta_1, K, C) \in A(x, k, p)$ that maximizes the expected value of his intertemporal utility function, given his initial endowment and durable good price (x, k, p)

$$V(x, k, p) = \sup_{(\Theta_1, K, C) \in A(x, k, p)} J^{\Theta_1, K, C}(x, k, p)$$

3. A semi-explicit solution for the no transaction costs problem

In this section we provide semi-explicit expressions for the optimal consumption flow of the perishable good and the optimal trading strategies for the financial assets and the durable good. Our result generalizes Framstad et al. (1999) who considered the optimal consumption and portfolio selection of an agent who consumes only a perishable consumption good, in a Lévy-driven financial asset market. It also extends the result of Damgaard et al. (2003), who analyzed the same problem in a market with both a perishable and a durable consumption good, but assuming that the risky financial assets prices follow a geometric Brownian motion (without jumps).

In this section we assume a framework where trading in the durable good involves no cost. Therefore, the agent's optimal perishable consumption and trading strategies are independent of the composition of his wealth (the fraction invested in the risky asset and in the durable good). Thus, we can eliminate our state variable K and obtain the optimal portfolio and consumption as a function of the two remaining state variables (X, P). The following theorem characterizes the optimal solution.

Theorem 1. *Suppose that*

$$\lim_{R \to \infty} E[e^{-\rho T_R} V(X_{T_R}, P_{T_R})] = 0$$

where $T_R = \min(R, \inf\{|X_t^*| \geq R\})$ *and* X_t^* *is the wealth process when the consumer chooses the optimal perishable consumption and trading strategy. Then the value function is given by*

$$V(x,p) = \frac{1}{1-\gamma}\alpha_v p^{-(1-\beta)(1-\gamma)} x^{(1-\gamma)}$$

and the optimal controls are $C_t = \alpha_c X_t^*$, $K_t = \alpha_k X_t^*/P_t$ *and* $\Theta_{1t} = \alpha_\theta X_t^*$, *where*

$$\alpha_k = \frac{1}{\gamma\sigma\sigma_{P1}}\left[-\gamma\alpha_\theta\sigma^2 + \mu - r - (1-\beta)(1-\gamma)\sigma\sigma_{P1} + \int_{-1}^{L} \eta[(1+\alpha_\theta\eta)^{-\gamma} - 1]\, dq(\eta)\right],$$

$$\alpha_c = \frac{\beta}{1-\beta}\alpha_k\left[(1-\beta)(1-\gamma)\left(\sigma_{p_1}^2 + \sigma_{P_2}^2\right) + \gamma\left(\alpha_\theta\sigma\sigma_{P_1} + \alpha_k^2\left(\sigma_{p_1}^2 + \sigma_{P_2}^2\right)\right) + r - \mu_P + \delta\right],$$

and

$$\alpha_v = \beta\alpha_c^{\beta(1-\gamma)-1}\alpha_k^{(1-\gamma)(1-\beta)}$$

where α_θ *is the solution to Eq.* (A.3) *given in the appendix, after replacing* α_k, α_c and α_v *by the expressions above.*

The optimal perishable consumption and trading strategy can be seen to consist of three components: the consumption of the perishable good C_t, the value $P_t K_t$ invested in the durable good and the value Θ_{1t} invested in the risky stock are constant fractions of the total wealth.

Note that, in the absence of jumps, the transformed Hamilton–Jacobi–Bellman equation (A.2) reduces to Eq. (A.4) in Damgaard et al. (2003). Thus our optimal solution in the absence of jumps would be the same as theirs. In the above solution, the change in the optimal level of durable good consumption, caused by the jumps, is ambiguous. It is affected by the term $\int_{-1}^{L} \eta[(1+\alpha_\theta\eta)^{-\gamma} - 1]\, dq(\eta)$, which is negative, and also by the decrease[3] in the optimal stock market investment, through the term $-\gamma\alpha_\theta\sigma^2$. In our simulations, it is clear that the second effect dominates the first one, and the optimal durable good consumption increases. Likewise, the optimal perishable good consumption also increases, because it varies in the same direction as the optimal durable good consumption, according to the solution α_c above.

4. Solution to the transaction costs problem

In this section we consider the optimal portfolio selection and consumption policies for an agent who faces positive transaction costs $\lambda > 0$. In this framework we cannot find an explicit solution for the consumer's problem. We characterize the optimal solution and then we find the optimal strategies using numerical simulations.

We show that, just as in Damgaard et al. (2003), the existence of fixed transaction costs implies that the durable good will be traded at most at a countable number of times. They have shown that whenever the consumer buys or sells the durable good, the ratio between a consumers' wealth and his durable good stock (prior to the transaction) is constant. That is, the consumer trades the durable good whenever this ratio reaches a given threshold.

We also characterize the stopping times at which the consumer trades the durable good but considering the presence of jumps. Unlike the model studied by the former authors, the ratio between an agent's wealth and his durable good stock immediately before the trade is not necessarily constant due to the existence of jumps in the process driving risky asset prices.

In order to proceed we first show some important properties of the value function, namely its boundness, monotonicity, continuity and homogeneity. First, we assume that the value function is finite and satisfies the dynamic programming equation. More formally we make the following.

Conjecture 2. *For all* (x,k,p) *belonging to the solvency region, an optimal policy exists, and the value function is finite and satisfies the dynamic programming principle. Therefore for all the stopping times* τ,

$$V(x,k,p) = \sup_{(\Theta_1,K,P) \in A(x,k,p)} E\left[\int_0^\tau e^{-\rho t} U(C_t, K_t)\, dt + e^{-\rho\tau} V(X_\tau, K_\tau, P_\tau)\right] \tag{1}$$

where $A(x,k,p)$ *represents the space of admissible controls for the initial endowment and durable good price* (x,k,p).

Note that for every t with $0 \leq t \leq \tau$, the investment in the risky financial asset must be such that a jump does not cause insolvency. That is, for every t with $0 \leq t \leq \tau$, $(X_{t^-} - \theta_1\eta, K_t, P_t)$ must belong to the solvency region *a.s.*, for every $\eta \in (-1, L)$.

For every time t before the first time that the durable good is traded, the durable good stock is given by $K_t = K_0 e^{-\delta t}$. Denoting by τ the first time that the durable good is traded, and applying (1), we get

$$V(x,k,p) = \sup_{(\Theta_1,K,P) \in A(x,k,p)} E\left[\int_0^\tau e^{-\rho t} U(C_t, ke^{-\delta t})\, dt + e^{-\rho\tau} V(X_{\tau^-} - \lambda k e^{-\delta\tau} P_\tau, K_\tau, P_\tau)\right]$$

where K_τ must be such that $(X_{\tau^-} - \lambda k e^{-\delta\tau} P_\tau, K_\tau, P_\tau)$ belongs to the solvency region.

[3] Jumps cause the fraction of wealth invested in the stock market to decrease. See, for example, Liu et al. (2003).

The next theorem presents some properties of the value function. We do not provide the proof of this theorem because it follows from Damgaard et al. (2003) with only minor modifications.

Theorem 3. *The value function $V(x,k,p)$ satisfies*

1. *For all (x,k,p) belonging to the solvency region*
$$\frac{1}{1-\gamma}\underline{\alpha} p^{-(1-\beta)(1-\gamma)}(x-\lambda kp)^{1-\gamma} \le V(x,k,p) \le \frac{1}{1-\gamma}\alpha_v p^{-(1-\beta)(1-\gamma)}x^{1-\gamma} \tag{2}$$
where α_v is given in Theorem 1, and
$$\underline{\alpha} = \frac{\beta^{\beta(1-\gamma)}(1-\beta)^{(1-\beta)(1-\gamma)}r^{\beta(1-\gamma)}}{\rho+\delta(1-\beta)(1-\gamma)}.$$
2. *For each $(k,p)\in\mathbb{R}^2_+$, $V(x,k,p)$ is strictly increasing and concave in x on the solvency region.*
3. *For each $(k,p)\in\mathbb{R}^2_+$, $V(x,k,p)$ is continuous in x on the solvency region.*
4. *$V(x,k,p)$ is homogeneous of degree $1-\gamma$ in (x,k) and of degree $\beta(1-\gamma)$ in (x,p) for all (x,k,p) belonging to the solvency region.*

Now, we will use the homogeneity of the value function to reduce the dimensionality of the problem. Note that
$$V(x,k,p) = k^{1-\gamma}p^{\beta(1-\gamma)}v(x/(kp)) \tag{3}$$
for every (x,k,p) belonging to the solvency region where $v(z) = V(z,1,1)$, and the solvency region is the closure of the set $z\in(\lambda,\infty)$. If $z=\lambda$, the agent must sell his entire stock of the durable good to avoid insolvency. Therefore $v(\lambda)=0$. Note also that because of Theorem 3, item 1, the transformed value function is also bounded, with
$$\frac{1}{1-\gamma}\underline{\alpha}(z-\lambda)^{1-\gamma} \le v(z) \le \frac{1}{1-\gamma}\alpha_v z^{1-\gamma}$$

Let $Z_t = X_t/(K_tP_t)$ denote the transformed wealth process and $\widehat{C}_t = C_t/(K_tP_t)$ and $\widehat{\Theta}_{1t} = \Theta_{1t}/(K_tP_t)$ denote the transformed controls for perishable consumption and risky asset investment, respectively. Note that the condition that we presented before, and that must be satisfied by Θ_{1t} to avoid bankruptcy, becomes, in the transformed model
$$\widehat{\Theta}_{1t}: \{Z_t - \widehat{\Theta}_{1t}\eta \ge \lambda\} \quad \text{for every } t:\ t\ge 0,\ \ \eta\in(-1,L) \text{ a.s.}$$

Using the dynamic programming principle and (3) we obtain the following equation:
$$p^{\beta(1-\gamma)}v(z) = \sup_{(\widehat{\Theta}_1,\widehat{C})\in A(z)} E\left\{\frac{1}{1-\gamma}\int_0^\tau e^{-\overline{\rho}t}\widehat{C}_t^{\beta(1-\gamma)}P_t^{\beta(1-\gamma)}\,dt + e^{-\rho\tau}P_\tau^{\beta(1-\gamma)}\frac{(Z_{\tau^-}-\lambda)^{1-\gamma}}{1-\gamma}M\right\}$$
where $p=P_0$, $z=Z_0$, $\overline{\rho}=\rho+\delta(1-\gamma)$, $A(z)$ is the set of admissible controls for initial transformed wealth z, τ is the first time when the agent trades the durable good,
$$M = \sup_{K_\tau \le ke^{-\delta\tau}(Z_{\tau^-}-\lambda)/\lambda}(1-\gamma)\left(\frac{ke^{-\delta\tau}(Z_{\tau^-}-\lambda)}{K_\tau}\right)^{\gamma-1} v\left(\frac{ke^{-\delta\tau}(Z_{\tau^-}-\lambda)}{K_\tau}\right) \tag{4}$$
and $k=K_0$.

This framework combines a stochastic control problem and an impulse control problem for a jump diffusion. Its solution may be approximated by the solution of a stochastic control problem combined with the solution of an optimal stopping problem,[4] where the terminal value function is given by $f(z)=((z-\lambda)/(1-\gamma))M$. Note that we can recover the optimal durable good stock immediately after a transaction from the following equality:
$$K_\tau = \frac{K_{\tau^-}(Z_{\tau^-}-\lambda)}{z^*}$$
where $z^* = \arg\max_{z\ge\lambda} z^{\gamma-1}v(z)$.

The Hamilton–Jacobi–Bellman (HJB) equation for the problem defined above is
$$0 = \max\left\{H(z,v,v',v''), \frac{(z-\lambda)^{1-\gamma}}{1-\gamma}M - v(z)\right\} \tag{5}$$

[4] See, for example, Oksendal and Sulem (2005).

where

$$H(z,v,v',v'') = \sup_{\widehat{\theta}_1,\widehat{c}\in A(z)} \left\{ \mathcal{L}^{\widehat{\theta}_1,\widehat{c}} v(z) + \frac{1}{1-\gamma}\widehat{c}^{\beta(1-\gamma)} - \overline{\rho} v(z) \right\} \tag{6}$$

and

$$\begin{aligned}\mathcal{L}^{\widehat{\theta}_1,\widehat{c}} v(z) = {} & v(z)\beta(1-\gamma)\left[\mu_P - \frac{1}{2}(1-\beta(1-\gamma))\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)\right] \\ & + v'(z)\left[r(z-1)+\widehat{\theta}_1(\mu-r)-\widehat{c}+\delta z-\mu_P(z-1)-\delta+(1-\beta(1-\gamma))[(z-1)(\sigma_{P_1}^2+\sigma_{P_2}^2)+\widehat{\theta}_1\sigma\sigma_{P_1}]\right] \\ & + v''(z)\left[\frac{1}{2}\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)(z-1)^2+\widehat{\theta}_1\sigma\sigma_{P_1}(1-z)+\frac{1}{2}\widehat{\theta}_1^2\sigma^2\right] \\ & + \int_{-1}^{L}[v(z+\widehat{\theta}_1\eta)-v(z)-v'(z)\widehat{\theta}_1\eta]\,dq(\eta)\end{aligned}$$

Unfortunately, due to the presence of transaction costs, we cannot guarantee that the transformed value function is sufficiently smooth, *i.e.*, the derivatives $v'(z)$ and $v''(z)$ may be not well-defined. In that case it would be difficult to characterize the function $H(z,v,v',v'')$ and to find a solution that satisfies the HJB equation in the classical sense above. The typical solution for these cases is to approach the HJB problem by a similar-class problem with a so-called viscosity solution.[5] It can be shown that all solutions of an HJB problem are solutions of a viscosity problem, but the reverse is not necessarily true. In that sense, the problem solved by the viscosity solution can be seen as a generalization of the HJB problem. We use this concept of viscosity solutions to show that the above HJB problem has a solution in this weaker sense. We will also prove that this solution is continuous on all the solvency region. Next, Theorem 4 shows that, under certain conditions, the viscosity solution is unique. Proofs are presented in the Appendix.

Theorem 4. *If $\rho > (1-\gamma)(\beta\mu_P - 1/2\beta(1-\beta(1-\gamma))(\sigma_{P_1}+\sigma_{P_2})^2-\delta)$ then the function $v(z)$ is a viscosity solution of the Hamilton–Jacobi–Bellman equation for the transformed problem and v is continuous in $[\lambda,\infty)$.*

Theorem 5. *Let us assume that*

1. $\rho > (1-\gamma)(\beta\mu_P - 1/2\beta(1-\beta(1-\gamma))(\sigma_{P_1}+\sigma_{P_2})^2-\delta)$;
2. $\underline{v}$ *and* $\overline{v}$ *are a continuous subsolution and supersolution, respectively, with sublinear growth in* $[\lambda,\infty)$, *and with* $\underline{v}(z) \geq (M/(1-\gamma))(z-\lambda)^{1-\gamma}$;
3. $\underline{v}(\lambda) = \overline{v}(\lambda) = 0$;
4. $\tilde{\theta} = \widehat{\theta}$, *where* $\tilde{\theta}$ *and* $\widehat{\theta}$ *(the optimal portfolio choices associated with the supersolution and subsolution, respectively) are given in the appendix.*

Then $\underline{v}(z) \leq \overline{v}(z)$.

Assumption 1 is required in order to assure that the solution is bounded. The assumption that $\underline{v}(z) \geq (M/(1-\gamma))(z-\lambda)^{1-\gamma}$ means that the value function, associated with the supersolution, is always larger or equal that the value the consumer would get if the traded the durable good immediately. Assumption 3 requires that the subsolution value must be zero, whenever the consumer is on the boundary of the solvency region. Note that, in this boundary, the consumer must sell the durable good immediately, in order to avoid insolvency, and thereafter his wealth, consumption and utility will be zero. Finally, assumption 4 is a technical requirement in order to apply the comparison principle to this problem.

Unicity of the viscosity solution follows from the theorem above together with the fact that $\underline{v}(z) \geq \overline{v}(z)$ by definition, and also because $v(z)$ has to be in between those bounding functions. Thus $v(z) = \underline{v}(z) = \overline{v}(z)$.

5. Characterization of the optimal consumption and trading strategies

In this section we show that, under specific conditions, the solvency region can be divided into three zones. In the first one it is optimal to sell the durable good, in the second the agent does not trade the durable consumption good, and in the last one the optimal trading strategy is to buy the durable good. As before, let z denote the ratio between the total wealth and the value of the stock of durable goods. The decision to sell the durable good happens when the ratio z is below a certain threshold. When there is the possibility of significant downward jumps in the stock market, there is a larger risk that in a given time interval the ratio z falls below the critical threshold. Hence, the possibility of large downward jumps tends to anticipate the sale of the durable good, implying an effective increase in the critical threshold for z at which the good is sold.

[5] See, for example, Crandall et al. (1992).

The next theorem shows that the no-trading region is the set

$$N = \{z > \lambda : v(z) > f(z)\}$$

where $f(z) = ((z-\lambda)^{1-\gamma}/(1-\gamma))M$ is the stopping reward function, where M reflects the value function just after the transaction of the durable good, just as defined in Section 4. We will also show that the no-trading region is an interval, delimited by $\overline{z}$ and $\underline{z}$, where $\overline{z} > \underline{z} > \lambda$. Whenever the agent is outside the no-trading region, transaction of the durable good takes place and the value of the state variable z immediately after the trade is reset to z^* located in the no-trading region N.

Theorem 6. *If $H(z, f, f', f'')$ is increasing in the second argument and the set $\{z: H(z, f, f', f'') > 0\}$ is an interval, then there exist numbers $\overline{z}$ and $\underline{z}$ with $\overline{z} > \underline{z} > \lambda$ such that $N = (\underline{z}, \overline{z})$,and z^*, defined above belongs to the no-trading region.*

The proof of Theorem 5 follows from Damgaard et al. (2003) with slight modifications.

Fig. 1 shows the buying, selling and no trading region of the risky asset in the (x, kp) space.

The no-trading region corresponds to the cone

$$N = \left\{(x, k, p) : \underline{z} < \frac{x}{kp} < \overline{z}\right\}$$

When x/kp attains the lower boundary of the no-trading region, the durable good stock becomes too high relative to the agent's total wealth. Therefore, he sells the durable good, and the state variable z value after the trade will equal z^*. If z gets out of the no-trading region through its upper boundary, the durable good stock becomes too low, and the agent buys the durable good until $z = z^*$. Note that the optimal trading strategy involves infrequent trading: the agent executes an initial durable good trade if z lies outside the no-trading zone, and then he only trades this good again once z attains one of the no-trading region boundaries.

It can be shown that the slope of the straight line representing the transaction in the (x, kp) space is $\lambda z^*/(z^* - z^- + \lambda)$, where z^- represents the state variable value immediately before the transaction. In Damgaard et al. (2003) this slope equals the constant $\lambda z^*/(z^* - \overline{z} + \lambda)$ when the agent buys the durable good and $\lambda z^*/(z^* - \underline{z} + \lambda)$ when he sells the durable good. z is continuous, and therefore, the agent trades the durable good at the stopping time $\{\tau : z_\tau = \underline{z} \text{ or } z_\tau = \overline{z}\}$. Note that, in our model, the risky asset price evolves according to a Lévy process, which implies that the state variable z may jump over any of the no-trading region boundaries. Therefore, he may trade the risky asset at a point such as A in the graph above, if the Brownian motion part drives the value of z to one of the boundaries, but he may also trade the durable good at a point such as B, if a jump in the risky asset price causes z to jump over any of the boundaries.

In the last part of this section we present the optimal perishable good consumption and the optimal risky asset investment allocation. Using Eq. (6) we have

$$c^* = \left(\frac{v'(z)}{\beta}\right)^{-1/(1-\beta(1-\gamma))}$$

and θ^* is the solution to

$$\begin{aligned} 0 &= v'(z)(\mu - r - (1-\gamma(1-\gamma))\sigma\sigma_{P1}) + v''(z)(\theta\sigma^2 - (z-1)\sigma\sigma_{P_1}) \\ &\quad + \int_{-1}^{L} [v'(z+\theta\eta)\eta - v'(z)\eta]\, dq(\eta) \end{aligned} \tag{7}$$

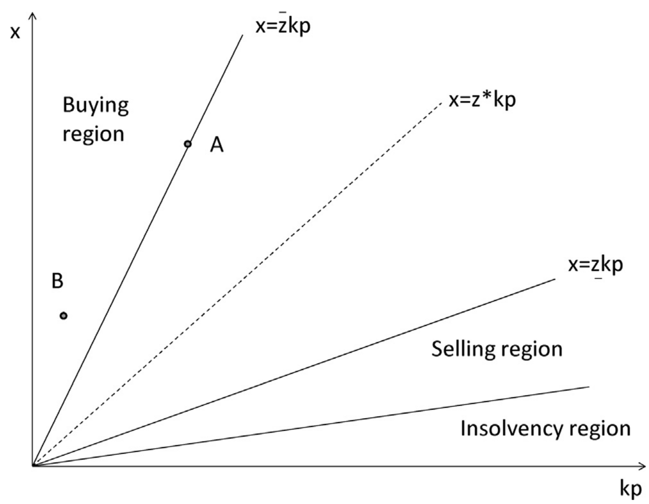

Fig. 1. Buying, selling and no-trading regions.

Note that in this equation there are two extra terms that are not present in the solution for a mean–variance optimizing investor in a Brownian motion context. The term $(z-1)\sigma\sigma_{P_1}$ is a hedging term that is related to the correlation between the risky asset price and the durable good price, and $\int_{-1}^{L}[v'(z+\theta\eta)\eta - v'(z)\eta]\, dq(\eta)$ derives from the existence of jumps in the risky asset price.

6. Numerical results

In this section we present the results of the numerical simulations for the transaction costs problems. A detailed description of the algorithm used is provided in Appendix B. In the numerical simulations we used the following parameter values for the baseline scenario.

Preferences: We followed Yao and Zhang (2005) and assumed that $\beta = 0.8$ and $\rho = 0.04$. We set the curvature parameter $\gamma = 0.9$.

Durable price process: We set the drift of the durable good price process at 2%.[6] Regarding the remaining parameters, we followed Yao and Zhang (2005), and assumed that the durable good price and stock market price processes are uncorrelated, $\sigma_{P_2} = 10\%$, $\delta = 1.5\%$, and $\lambda = 6\%$.

Asset price processes: Once again we followed Yao and Zhang (2005) and set the drift and Brownian volatility of the stock market process at 4% and 15.7%, respectively. We considered large and infrequent jumps and, as in Liu et al. (2003), in our baseline scenario we assumed that there is a 4% probability of a -25% jump each year.

In the following figures we will compare our baseline scenario that involves jumps in the risky asset price process, with an alternative framework, in which there are no risky asset price jumps. When jumps were considered, we adjusted the values of the drift and Brownian volatility of the stock market price process, in order to maintain the first two moments of the distribution unchanged.

Fig. 2 displays the difference between the value function and its value if a durable asset transaction were performed $(v(z) - (M/(1-\gamma))(z-\lambda)^{1-\gamma})$. Regarding the difference between our scenario with jumps in the risky asset price and the alternative scenario with no jumps, we observe that with the existence of jumps the no-trading region becomes narrower (with jumps $\underline{z} = 0.88$, $z^* = 3.56$ and $\overline{z} = 9.78$, and without jumps $\underline{z} = 0.87$, $z^* = 3.68$ and $\overline{z} = 10.34$). However, note that the change in the no-trading region is asymmetric – the lower boundary remains almost unchanged and the upper boundary decreases substantially.

When the consumer chooses the boundaries of the no-trading region, he has to weigh the benefit of trading the durable good (that is, the benefit of moving to a preferred wealth-to-durable-good ratio) against the expected cost of trading it. Jumps cause both direct and indirect effects on these costs and benefits. First, jumps may lead to sudden and very unfavorable changes in the wealth-to-durable-good ratio. Therefore, the consumer would like to increase the lower boundary of the no-trading region, in order to avoid a very low wealth-to-durable-good, following a jump. Second, jumps cause a decrease in the fraction of wealth invested in the stock market (see Fig. 4), which leads to a decrease in the volatility of wealth and the expected number durable good trades, which implies that the consumer would like to decrease the size of the no-trading region, to rebalance the expected cost and benefit of trading the durable good. Lastly, jumps cause the expected rate of change of the wealth-to-durable-good ratio to decrease, because the consumer invests a smaller fraction of his wealth in the stock market (see Fig. 4) and consumes a higher fraction of his wealth (see Fig. 5). Therefore, for given initial wealth-to-durable good ratio and no-trading region boundaries, the expected time until the upper (lower) boundary is attained increases (decreases), and the consumer would like to shift the no-trading region to the left to rebalance the expected times until the next trade.

Note that all the effects lead to a lower value of the upper boundary of the no-trading region, but the impact on the lower boundary is ambiguous, because the sign of the effect of the change in the wealth-to-durable-good ratio drift, counteract the remaining to effects, on the lower boundary. The change in the no-trading region shape remains asymmetric, even when the jumps are symmetric, as we will show in the next section.

Fig. 3 displays the agent's relative risk aversion. Relative risk aversion is only slightly affected by the existence of jumps. Its pattern is very similar in the scenarios with jumps and without jumps. It is low near the boundaries of the no-trading region because the utility loss from trading the durable good is low near $\underline{z}$ and $\overline{z}$ (see Fig. 1), and it increases as z approaches the middle of the no-trading region. The agent behaves in a more risk averse manner z^* because he wants to avoid the cost of trading the durable good. Near the boundaries, the loss from these transaction cost is partially compensated by the benefit derived from the change to the optimal wealth to durable good ratio z^*.

Fig. 4 shows the fraction of wealth invested in the risky asset.[7]. As in Liu et al. (2003) jumps cause a substantial decrease in the fraction of wealth invested in the stock market (approximately -16%). Jumps make extreme stock returns more likely (the returns distribution becomes leptokurtic and skewed to the left), which drives the fraction of wealth invested in the stock market lower.

[6] The value that we have chosen is compatible with the real housing returns, for the 12 largest Metropolitan Statistical Areas (MSA), estimated by Goetzmann and Spiegel (2000) (annualized geometric mean return between -1.1% and 3.26&).

[7] Note that $\widehat{\theta}/z = \theta/x$. Therefore we can easily calculate the fraction of wealth invested in the risky asset from the variables of the transformed model.

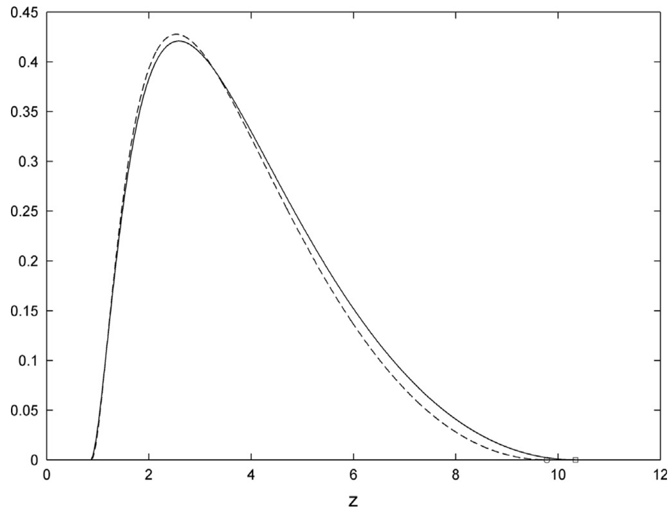

Fig. 2. The dashed (solid) line represents the difference between the value function and its value if the consumer decided to trade the durable good immediately, for the baseline scenario (no jump scenario). The circle (rectangle) shows the upper boundary of the no-trading region for the baseline scenario (no jump scenario).

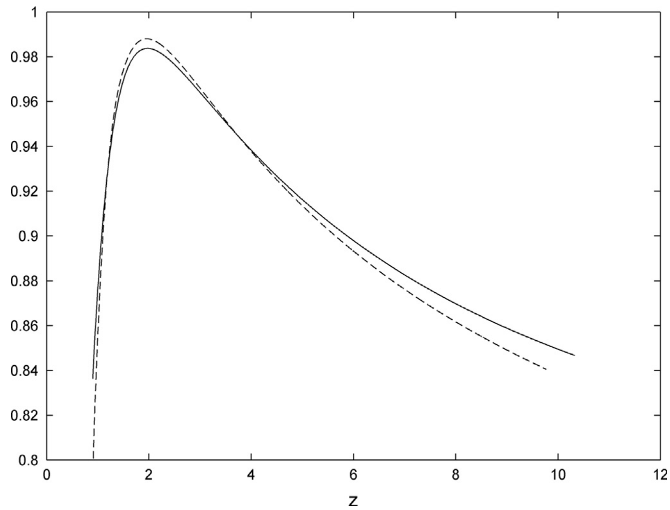

Fig. 3. The dashed (solid) line represents the relative risk aversion, as a function of z, for the baseline (no jump) scenario.

Fig. 5 displays the optimal propensity to consume as a function of z. As shown in Damgaard et al. (2003) the optimal propensity to consume is increasing in z whenever the relative risk aversion with respect to wealth changes is higher than the "relative risk aversion" with respect to consumption changes (i.e. $-zv''(z)/v'(z) > -c(\partial^2 U/\partial c)/(\partial U/\partial c) = 1-\beta(1-\gamma)$). In our base scenario, $1-\beta(1-\gamma) = 0.92$. Therefore, the optimal propensity to consume is decreasing for low and high wealth-to-durable-good ratios, and is increasing for the remaining values of z (see Figs. 3 and 5). Note also that an agent that faces a risky asset price with jump consumes a smaller fraction of wealth than an agent that can invest in a risky asset with no price jumps, for each z. This pattern is a consequence of the fact that jumps increase the risk of investing in the stock market, and consequently, this asset becomes a less effective vehicle to transfer consumption to the future, and the agent chooses to increase his present consumption.

7. Parameter sensitivity

In this section we consider the impact of changes in the jump size and probabilities on the no-trading region, the average propensity to consume the perishable good, and the stock market investment pattern. In the first line of Table 1 we present the baseline scenario ($J^D = -25\%, P^D = 4\%, J^U = 0, P^U = 0$). Even though jumps are unlikely, in this scenario, their size is high, which causes a substantial decrease in the fraction of wealth invested in the stock market, an increase in the average propensity to consume, and a decrease in the no-trading region (mainly in the upper boundary).

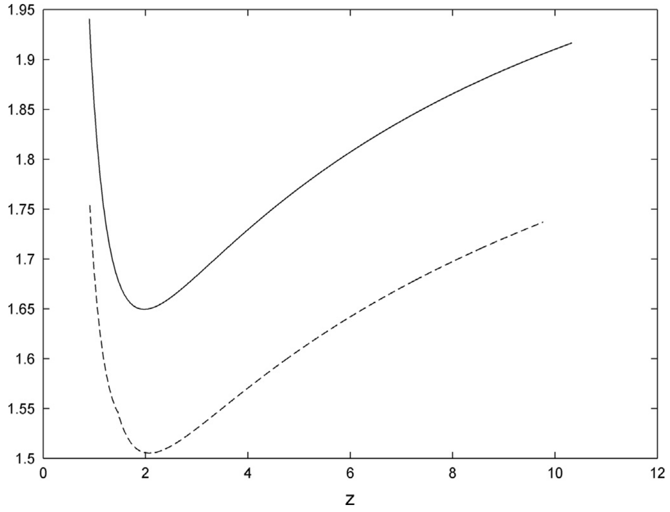

Fig. 4. The dashed (solid) line represents the fraction of the consumer's wealth invested in the stock market, plotted against z, for the baseline (no jump) scenario.

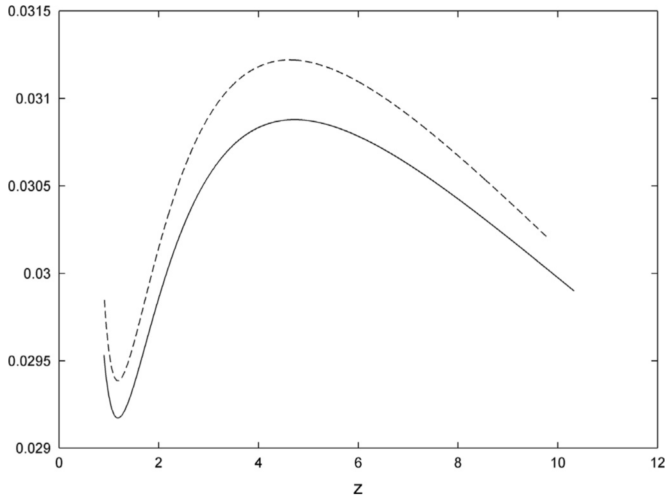

Fig. 5. The perishable consumption rate is plotted against z, in the baseline scenario (dashed line) and in the no jump scenario (solid line).

Table 1
Sensitivity analysis.

Baseline scenario	$\Delta\underline{z}$	Δz^*	$\Delta\overline{z}$	Δc (%)	$\Delta(\theta/x)$ (%)
	0.01	−0.12	−0.56	0.029	−16.05
$J^D=-18.75\%,\ P^D=4\%,\ J^U=0,\ P^U=0$	0.01	−0.05	−0.35	0.015	−7.39
$J^D=-31.25\%,\ P^D=4\%,\ J^U=0,\ P^U=0$	0.02	−0.23	−1.02	0.056	−27.75
$J^D=-25\%,\ P^D=2\%,\ J^U=0,\ P^U=0$	0.01	−0.07	−0.34	0.015	−8.73
$J^D=-25\%,\ P^D=8\%,\ J^U=0,\ P^U=0$	0.04	−0.21	−0.99	0.054	−27.98
$J^D=-25\%,\ P^D=4\%,\ J^U=25\%,\ P^U=4\%$	0.02	−0.16	−0.81	0.041	−16.9
$J^D=-25\%,\ P^D=4\%,\ J^U=12.5\%,\ P^U=4\%$	0.01	−0.13	−0.65	0.032	−16.59
$J^D=-25\%,\ P^D=4\%,\ J^U=25\%,\ P^U=2\%$	0.01	−0.13	−0.68	0.035	−16.36

This table displays the changes in the no-trading region boundaries ($\underline{z}$ and $\overline{z}$), the optimal wealth-to-durable-good ratio after a trade (z^*), the average propensity to consume and the average fraction of wealth invested in the stock market, relative to the no jump scenario. $J^D(J^U)$ represents the size of the negative (positive) jump and $P^D(P^U)$ represents the probability of a negative (positive) jump.

The next two lines show the effect of changing the jump size (25% increase, in the second line and 25% decrease in the third line). Note that a 25% increase in the jump size causes a substantial decrease in the fraction of wealth invested in the stock market and in the upper boundary of the no-trading region. The lower boundary remains almost unchanged, because the decrease in the drift of the wealth-to-durable-good process, that is caused by the decrease in the fraction of wealth invested in the stock market, partially counteracts the remaining effects that would lead to an increase in the lower boundary (see previous section).

Lines 4 and 5 show the effects of halving and doubling the jump probability. Note that the impact of doubling (halving) the jump probability is approximately the same as the impact of a 25% increase (decrease) in the jump size.

The remaining lines display the effects on the optimal consumer choices of considering both positive and negative jumps. In line 6 we consider both positive and negative jumps, with equal absolute size and probability. The main effect of including positive jumps in the model is to further decrease the upper boundary of the no-trading region, relative to the baseline scenario. The lower boundary and the fraction of wealth invested in the stock market are almost the same.

It is also interesting to compare lines 5 and 6 of Table 1. Note that, in both cases, the probabilities and absolute sizes of the jumps are the same, but in line 5 the distribution of jumps is asymmetric, while in line 6 it is symmetric. Clearly, the fraction of wealth invested in stocks is more sensitive to negative jumps. Curiously, the upper boundary of the no-trading region decreases more, when all the jumps are negative, because the indirect effect of the decrease in the Brownian volatility and drift of the wealth process, which is higher when all the jumps are negative, seems to be more important than the direct impact of positive jumps on the upper boundary.

Note also that the asymmetric character of the change in the no-trading region is not driven by the asymmetric distribution of jumps. The no-trading region changes remain asymmetric, even when the distribution of jumps is symmetric (line 6), because the decrease in the fraction of wealth invested in stocks decreases the drift of the wealth-to-durable-good process.

8. Conclusion

In this paper we studied the problem of an agent who faces a geometric Lévy stock market price process and consumes both a durable and perishable good. This generates a return distribution with a higher variance than the distribution in the no-jump case. This increase in the stock market risk implies a narrower no-trading region, a higher current consumption and a lower investment in the stock market.

We showed that, in the presence of durable good transaction costs, the agent must not trade the durable good continuously, in order to avoid insolvency. The solvency region is divided into three zones: a buying region, a selling region and a no-trading region. Whenever the ratio of wealth-to-durable good value becomes sufficiently low, the agent sells the durable good; whenever that ratio is too high, he buys the good. In Damgaard et al. (2003), where no jumps in the stock market are present, the agent only trades the durable good exactly at the boundary between the no-trading and the buying region or at the boundary between the no-trading and the selling region. In our framework, a jump in the stock market may cause a sudden change in consumer's wealth and as a consequence, a sudden change in the wealth-to-durable good value ratio. Therefore, the wealth-to-durable good ratio may cross any of the boundaries of the no-trading region after a jump, even if in the previous moment this ratio was strictly inside the no-trading region, leading the consumer to trade the durable good. Hence, on one hand the agent would prefer to shrink the no-trading region in order to reduce the risk of finding himself outside the interval due to the jumps, but on the other hand, there are also indirect effects on the shape and size of the no-trading region, due to changes in the optimal perishable good consumption and stock market investment. The combination of these two kinds of effects determines the optimal shape of the no-trading region.

We also show that the change in the no-trading region occurs, mainly, in the upper boundary, even when the jump distribution is symmetric. The decrease in drift of the wealth-to-durable-good ratio process, that is caused by the decrease in stock market investment and the increase in perishable good consumption, is the main responsible for this asymmetric behavior.

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Appendix A

A.1. Proof of Theorem 1

The value function of the no transaction costs problem is

$$V(x,p) = \sup_{(\Theta_1, K, C) \in A(x,k,p)} E\left[\int_0^\infty e^{-\rho t} U(C_t, K_t)\, dt\right]$$

The Hamilton–Jacobi–Bellman equation for this problem is

$$\rho V(x,p)=\sup_{\theta_1,c,k\in\mathbb{R}^+}\left\{\begin{array}{c}\frac{1}{1-\gamma}(c^\beta k^{(1-\beta)})^{1-\gamma}+\left(r(x-kp)+\theta_1(\mu-r)+(\mu_P-\delta)kp-c\right)\frac{\partial V}{\partial x}\\ +\frac{1}{2}\left(\theta_1^2\sigma^2+k^2p^2\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)+2\theta_1 kp\sigma\sigma_{P1}\right)\frac{\partial^2 V}{\partial x^2}+\mu_P p\frac{\partial V}{\partial p}\\ +\frac{1}{2}p^2\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)\frac{\partial^2 V}{\partial p^2}+p\left(\sigma_{P_1}\left(\theta_1\sigma+kp\sigma_{P_1}\right)+kp\sigma_{P_2}^2\right)\frac{\partial^2 V}{\partial x\partial p}\\ +\int_{-1}^{L}\left[V(x+\eta\theta_1,p)-V(x,p)-\frac{\partial V}{\partial x}\theta_1\eta\right]dq(\eta)\end{array}\right\} \tag{A.1}$$

We follow Damgaard et al. (2003) and use the homogeneity of the instantaneous utility function to reduce the dimensionality of the problem. Note that, for $\kappa>0$, (Θ_1,K,C) is admissible with initial wealth x and initial durable price p, if and only if $(\kappa\Theta_1,K,\kappa C)$ is admissible with initial wealth κx and initial durable price κp. Since $U(\kappa C,K)=\kappa^{\beta(1-\gamma)}U(C,K)$ it follows that $V(\kappa x,\kappa p)=\kappa^{\beta(1-\gamma)}V(x,p)$, and in particular $V(x,p)=p^{\beta(1-\gamma)}V(x/p,1)=p^{\beta(1-\gamma)}v(x/p)$. Substituting this equality in (A.1) and simplifying we obtain

$$0=-v(y)\left[\rho-\beta(1-\gamma)\mu_P+\frac{1}{2}\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)\beta(1-\gamma)[1-\beta(1-\gamma)]\right]$$
$$+\sup_{\hat\theta_1,k,\hat c\in\mathbb{R}_+}\left\{\begin{array}{c}\frac{1}{2}v''(y)\left[\hat\theta_1^2\sigma^2+\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)(y-k)^2+2\hat\theta_1\sigma\sigma_{P_1}(k-y)\right]\\ +v'(y)\left[\begin{array}{c}(r+(1-\beta(1-\gamma))(\sigma_{P_1}^2+\sigma_{P_2}^2)-\mu_P)(y-k)-\delta k\\ +\hat\theta_1(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P_1})-\hat c\end{array}\right]\\ +\int_{-1}^{L}\left[v\left(y+\hat\theta_1\eta\right)-v(y)-v'\hat\theta_1\eta\right]dq(\eta)+\frac{1}{1-\gamma}(\hat c^\beta k^{1-\beta})^{1-\gamma}\end{array}\right\} \tag{A.2}$$

where $\hat c=c/p$ and $\hat\theta_1=\theta_1/p$. We claim that this equation as a solution of the form

$$v(y)=\frac{1}{1-\gamma}\alpha_v y^{1-\gamma}$$

with the maximizing controls $\hat c=\alpha_c y$, $\hat\theta_1=\alpha_\theta y$ and $k=\alpha_k y$. Substituting the previous equation in (A.2), we obtain

$$0=-\frac{1}{1-\gamma}\alpha_v y^{1-\gamma}\left[\rho-\beta(1-\gamma)\mu_P+\frac{1}{2}\beta(1-\gamma)(1-\beta(1-\gamma))\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)\right]$$
$$+\sup_{\hat\theta_1,k,\hat c\in\mathbb{R}_+}\left\{\begin{array}{c}-\frac{1}{2}\gamma\alpha_v y^{-\gamma-1}\left[\hat\theta_1^2\sigma^2+\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)(y-k)^2+2\hat\theta_1\sigma\sigma_{P_1}(k-y)\right]\\ +\alpha_v y^{-\gamma}\left[\begin{array}{c}(r+(1-\beta(1-\gamma))(\sigma_{P_1}^2+\sigma_{P_2}^2)-\mu_P)(y-k)-\delta k\\ +\hat\theta_1(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P_1})-\hat c\end{array}\right]\\ +\int_{-1}^{L}\left[\frac{\alpha_v}{1-\gamma}(y+\hat\theta_1\eta)^{1-\gamma}-\frac{\alpha_v}{1-\gamma}y^{1-\gamma}-\alpha_v y^{-\gamma}\hat\theta_1\eta\right]dq(\eta)+\frac{1}{1-\gamma}(\hat c^\beta k^{1-\beta})^{1-\gamma}\end{array}\right\}.$$

Ignoring the positivity constraints, the first order conditions for the maximizing controls are

$$0=\beta(\alpha_c y)^{\beta(1-\gamma)-1}(\alpha_k y)^{(1-\beta)(1-\gamma)}-\alpha_v y^{-\gamma} \tag{A.3}$$

$$0=\gamma\alpha_v y^{-\gamma-1}(\sigma\sigma_{P_1}(y-k)-\hat\theta_1\sigma^2 y)+\alpha_v y^{-\gamma}(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P1})$$
$$+\int_{-1}^{\infty}[\alpha_v(y+\hat\theta_1\eta)^{-\gamma}\eta-\alpha_v y^{-\gamma}\eta]\,dq(\eta)$$
$$0=(1-\beta)\hat c^{\beta(1-\gamma)}k^{(1-\beta)(1-\gamma)-1}+\gamma\alpha_v y^{-\gamma-1}[(\sigma_{P_1}^2+\sigma_{P_2}^2)(y-k)-\hat\theta_1\sigma\sigma_{P_1}]$$
$$-\alpha_v y^{-\gamma}[r+(1-\beta(1-\gamma))(\sigma_{P_1}^2+\sigma_{P_2}^2)-\mu_p+\delta]$$

Using the first order conditions and the assumed control specification we get

$$\alpha_v=\beta\alpha_c^{\beta(1-\gamma)-1}\alpha_k^{(1-\beta)(1-\gamma)}$$
$$\alpha_c=\frac{\beta}{1-\beta}\alpha_k\left[(1-\beta)(1-\gamma)\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)+\gamma\alpha_\theta\sigma\sigma_{P1}+r-\mu_P+\delta+\gamma\alpha_k\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)\right]$$
$$\alpha_k=\frac{1}{\gamma\sigma\sigma_{P_1}}\left[-\gamma\alpha_\theta\sigma^2+\mu-r-(1-\beta)(1-\gamma)\sigma\sigma_{P1}+\int_{-1}^{L}\eta[(1+\alpha_\theta\eta)^{-\gamma}-1]\,dq(\eta)\right]$$

Using the optimal controls above, substituting in (A.2) and solving the equation, we obtain the optimal control α_θ.

A.2. Proof of Theorem 4

We will start this proof by presenting a definition of viscosity solution for second order partial differential equation in the context of our model. Then, we will prove that v is a viscosity subsolution and, finally, we will prove that it is also a supersolution of the HJB equation.

Definition 7. A locally bounded function $v \in USC[\lambda, \infty)$ is a viscosity subsolution of the HJB equation for the transformed problem if and only if, for every $\phi \in C^2([\lambda, \infty))$, and every global maximum point $z_0 \in [\lambda, \infty)$ of the function $v-\phi$

$$\max\left\{ H(z_0, v, \phi', \phi''), \frac{(z_0-\lambda)^{1-\gamma}}{1-\gamma} M - v(z_0) \right\} \geq 0$$

A locally bounded function $v \in LSC[\lambda, \infty)$ is a viscosity supersolution of the HJB equation for the transformed problem if and only if, for every $\phi \in C^2([\lambda, \infty))$, and every global minimum point $z_0 \in [\lambda, \infty)$ of the function $v-\phi$

$$\max\left\{ H(z_0, v, \phi', \phi''), \frac{(z_0-\lambda)^{1-\gamma}}{1-\gamma} M - v(z_0) \right\} \leq 0$$

A continuous function $v: [\lambda, \infty) \to \mathbb{R}$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

A.2.1. v is a viscosity subsolution

We will prove this result by contradiction. Let us assume that $v(z_0) = \phi(z_0)$ and $v(z) \leq \phi(z)$ for every $z \in B(z_0, \psi_1)$, where $B(z_0, \psi_1)$ represents the ball centered at z_0 with radius ψ_1. If v is not a subsolution then $H(z_0, v, \phi', \phi'') \leq -\varepsilon_1$ and

$$\frac{M}{1-\gamma}(z_0-\lambda)^{1-\gamma} - v(z_0) \leq -\varepsilon_1$$

for some $\varepsilon_1 > 0$. Given that $\phi \in C^2([\lambda, \infty))$, there is a $B(z_0, \psi_2)$ such that, for all $z \in B(z_0, \psi_2)$ we have $H(z, v, \phi', \phi'') \leq -\varepsilon_1$ and

$$\frac{M}{1-\gamma}(z-\lambda)^{1-\gamma} - v(z_0) \leq -\varepsilon_1$$

Let us define t_1 as the first jumping time of the Lévy process. Then, given that this process is right continuous, either $t_1 = 0$, *a.s.*, or $t_1 > 0$, *a.s.* For the latter hypothesis define the stopping-time τ_0 as

$$\tau_0 = \inf\{t_1, \inf t : z \notin B(z_0, \psi_2), \inf t : z > R\}$$

for a positive constant $R > z_0$. Using Itô's lemma we get

$$E[e^{-\overline{\rho}\tau_0} P_{\tau_0}^{\beta(1-\gamma)} \phi(z_{\tau_0})] = P_0^{\beta(1-\gamma)} \phi(z_0) + E\left[\int_0^{\tau_0} e^{-\overline{\rho}t} P_t^{\beta(1-\gamma)} (\mathcal{L}\phi(Z_t^*) - \overline{\rho}\phi(Z_t^*))\, dt\right],$$

where Z_t^* represents the value of the state variable Z at time t when the optimal controls are used. Therefore

$$\begin{aligned} E\left[e^{-\overline{\rho}\tau_0} P_{\tau_0}^{\beta(1-\gamma)} \phi\left(z_{\tau_0}\right)\right] &\leq P_0^{\beta(1-\gamma)} \phi(z_0) + E\left[\int_0^{\tau_0} e^{-\overline{\rho}t} P_t^{\beta(1-\gamma)} \left(H\left(Z_t^*, \phi, \phi', \phi''\right) - \frac{\widehat{C}_t^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] \\ &\leq P_0^{\beta(1-\gamma)} \phi(z_0) + E\left[\int_0^{\tau_0} e^{-\overline{\rho}t} P_t^{\beta(1-\gamma)} \left(-\varepsilon_1 - \frac{\widehat{C}_t^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] \end{aligned}$$

Defining $\kappa = E[\int_0^{\tau_0} e^{-\overline{\rho}t} P_t^{\beta(1-\gamma)}(\varepsilon_1)\, dt] > 0$, and using the fact that $v(z_0) = \phi(z_0)$ and $v(z_{\tau_0}) \leq \phi(z_{\tau_0})$, we have

$$P_0^{\beta(1-\gamma)} v(z_0) > E\left[e^{-\overline{\rho}\tau_0} P_{\tau_0}^{\beta(1-\gamma)} v\left(z_{\tau_0}\right) + \int_0^{\tau_0} e^{-\overline{\rho}t} P_t^{\beta(1-\gamma)} \left(\frac{\widehat{C}_t^{\beta(1-\gamma)}}{1-\gamma} \right) dt \right] + \kappa$$

which violates the dynamic programming principle.

Now let us consider the case $t_1 = 0$ *a.s.* If the jump causes the agent to trade the durable good, then using the dynamic programming principle with $t=0$ we have

$$P_0^{\beta(1-\gamma)} v(z_0) = P_0^{\beta(1-\gamma)} \frac{M}{1-\gamma}(z_0-\lambda)^{1-\gamma} \leq P_0^{\beta(1-\gamma)} (z_0-\lambda)^{1-\gamma} (z^*)^{\gamma-1} \phi\left(z^*\right)$$

where $z^* = \arg\max\{z^{\gamma-1}\phi(z)\}$.

A.2.2. v is a viscosity supersolution

We must prove that

$$0 \geq H(z_0, v, \phi', \phi'') \tag{A.4}$$

and

$$0 \geq \frac{M}{1-\gamma}(z_0-\lambda)^{1-\gamma} - v(z_0) \tag{A.5}$$

Inequality (A.5) always holds because the agent can always trade the durable good. We must prove that (A.4) also holds. Let τ_ζ denote the first exit time from the closed ball $N(\zeta, z_0)$ centered at z_0, and strictly contained in the solvency region. Let $(\widehat{\Theta}_{1t}, \widehat{C}_t)$ be an admissible policy with $(\widehat{\Theta}_{1t}, \widehat{C}_t) = (\theta_1, c)$ for $t \in [0, \tau_\zeta]$. Using the dynamic programming principle we have

$$P_0^{\beta(1-\gamma)} v(z_0) \geq E\left[\int_0^{h \wedge \tau_\zeta} e^{-\overline{\rho} t} P_t^{\beta(1-\gamma)} \left(\frac{c^{(1-\gamma)}}{1-\gamma}\right) dt + e^{-\overline{\rho}\tau_\zeta} P_{\tau_\zeta}^{\beta(1-\gamma)} v\left(z_{\tau_\zeta}\right)\right] \tag{A.6}$$

Using Itô's lemma and taking expectations

$$P_0^{\beta(1-\gamma)} \phi(z_0) = E\left[e^{-\overline{\rho}\tau_\zeta} P_{\tau_\zeta}^{\beta(1-\gamma)} \phi(z_{\tau_\zeta}) - \int_0^{h \wedge \tau_\zeta} e^{-\overline{\rho}_t} P_t^{\beta(1-\gamma)} (\mathcal{L}\phi(Z_t) - \overline{\rho}\phi(Z_t))\, dt\right] \tag{A.7}$$

Considering that $v(z_0) = \phi(z_0)$ and $v(z_{\tau_0}) \geq \phi(z_{\tau_0})$, and using (A.6) and (A.7)

$$P_0^{\beta(1-\gamma)} \phi(z_0) \geq E\left[\int_0^{h \wedge \tau_\zeta} e^{-\overline{\rho} t} P_t^{\beta(1-\gamma)} \left(\frac{c^{\beta(1-\gamma)}}{1-\gamma}\right) dt + e^{-\overline{\rho}\tau_\zeta} P_{\tau_\zeta}^{\beta(1-\gamma)} \phi\left(z_{\tau_\zeta}\right)\right]$$

$$= P_0^{\beta(1-\gamma)} \phi(z_0) + E\begin{bmatrix} \int_0^{h \wedge \tau_\zeta} e^{-\overline{\rho}_t} P_t^{\beta(1-\gamma)} (\mathcal{L}\phi(Z_t) - \overline{\rho}\phi(Z_t))\, dt \\ + \int_0^{h \wedge \tau_\zeta} e^{-\overline{\rho} t} P_t^{\beta(1-\gamma)} \left(\dfrac{c^{\beta(1-\gamma)}}{1-\gamma}\right) dt \end{bmatrix}$$

Therefore

$$0 \geq E\left[\int_0^{h \wedge \tau_\zeta} e^{-\overline{\rho}_t} P_t^{\beta(1-\gamma)} \left(\frac{c^{\beta(1-\gamma)}}{1-\gamma} + \mathcal{L}\phi(Z_t) - \overline{\rho}\phi(Z_t)\right) dt\right]$$

Now, letting $h \to 0$ we get

$$0 \geq \frac{c^{\beta(1-\gamma)}}{1-\gamma} + \mathcal{L}\phi(Z_t) - \overline{\rho}\phi(Z_t)$$

We know that the value function is continuous in (λ, ∞), because a concave function is continuous on the interior of its domain. Therefore, we are just left to prove that the value function is continuous at the boundary. We will base our proof in the following lemma.

Lemma 8. *For any $\lambda > 0$, there exist constants $\xi > 0$, $0 < \kappa \leq \lambda + \xi$ and $\alpha > 0$, such that*

$$V(x,k,p) \leq \psi(x,k,p) \equiv \frac{\alpha p^{-(1-\beta)(1-\gamma)}}{1-\gamma}(x - \lambda k p)^{1-\gamma} \quad \text{for every } (x,k,p) \in I_\xi \cup \Gamma_\kappa$$

where

$$I_\xi = \{((x,k,p)) \in \mathbb{R}^3_{++} : \lambda k p \leq x < (\lambda+\xi) k p\} \quad \text{and} \quad \Gamma_\kappa = \{(x,k,p) \in \mathbb{R}^3_{++} : kp \leq 1, \lambda k p \leq x < \kappa\}$$

Note that this result implies continuity of the value function at the boundary. If $x = \lambda k p$ the only admissible strategy is to liquidate the stock of durable good, which implies $V = 0$. On the other hand, note also that $\psi(x,k,p) \to 0$, when $(x,k,p) \to (x_0, k_0, p_0)$ through $I_\xi \cup \Gamma_\kappa$, where (x_0, k_0, p_0) lies in the boundary of the solvency region. Then, we get $0 \leq V(x,k,p) \leq \psi(x,k,p) \to 0$, for $(x,k,p) \to (x_0,k_0,p_0)$.

Proof. Let $Y_t = (X_t, K_t, P_t)$ and $y = (x,k,p)$ and define

$$F\left(y, \psi, D\psi, D^2\psi\right) = \sup_{(c,\theta)} \{A^{c,\theta}\psi(y) + U(c,k) - \rho\psi(y)\}$$

$$= \sup_{(c,\theta)} \left\{ \begin{array}{c} \frac{\partial \psi}{\partial x}[r(x-\lambda k p) + \theta(\mu - r) + (\mu_P - \delta) k p - c] + \frac{\partial \psi}{\partial k}\delta k + \frac{\partial \psi}{\partial p}\mu_P p + \frac{1}{2}\frac{\partial^2 \psi}{\partial p^2}\left(\sigma_{P_1}^2 + \sigma_{P_2}^2\right)p^2 \\ + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}\left[\theta^2\sigma^2 + k^2 p^2\left(\sigma_{P_1}^2 + \sigma_{P_2}^2\right) + 2\theta\sigma\sigma_{P_1} k p\right] + \frac{\partial^2 \psi}{\partial x\, \partial p} p\left[\sigma_{P_1}\left(\theta\sigma + k p \sigma_{P_1}\right) + \sigma_{P_2}^2 k\right] \\ + \int_{-1}^{L}\left[\psi(x+\theta\eta, k, p) - \psi(x,k,p) - \frac{\partial \psi}{\partial \psi}\theta\eta\right] dq(\eta) + \frac{1}{1-\gamma} c^{\beta(1-\gamma)} k^{(1-\beta)(1-\gamma)} - \rho\psi(x,k,p) \end{array} \right\}$$

Let us define the function $\phi(z)=\alpha(z-\lambda)^{1-\gamma}/(1-\gamma)$ where α is a constant. Note that $\psi(x,k,p)=k^{1-\gamma}p^{\beta(1-\gamma)}\phi(z)$. Then substituting this relation and the optimal c control in the previous equation

$$F=k^{1-\gamma}p^{\beta(1-\gamma)}\sup_{\theta}[\alpha(z-\lambda)^{-\gamma-1}G]$$

where

$$G=\left[\begin{array}{c} (z-\lambda)^2\left[-\delta+\beta\mu_P-\frac{1}{2}(1-\beta(1-\gamma))\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)-\frac{\rho}{1-\gamma}\right] \\ +(z-\lambda)\left[\begin{array}{c} r(z-1)+\theta(\mu-r)-(\mu_P-\delta)(z-1) \\ +(1-\beta(1-\gamma))[(z-1)(\sigma_{P_1}^2+\sigma_{P_2}^2)-\theta\sigma\sigma_{P_1}] \end{array}\right] \\ -\gamma\left[\frac{(1-z)^2}{2}\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)+(1-z)\theta\sigma\sigma_{P_1}+\frac{\theta^2\sigma^2}{2}\right] \\ +(z-\lambda)^{\gamma/(1-\beta(1-\gamma))}\left[\frac{1}{1-\gamma}-\left(\frac{\alpha}{\beta}\right)^{-1/(1-\beta(1-\gamma))}\right] \\ +\int_{-1}^{\infty}\left[\frac{\alpha}{1-\gamma}\left[(z+\theta\eta-\lambda)^{1-\gamma}-(z-\lambda)^{1-\gamma}\right]-\gamma(z-\lambda)^{-\gamma}\theta\eta\right]dq(\eta) \end{array}\right]$$

By the mean value theorem

$$\begin{aligned}&\int_{-1}^{\infty}\left[\frac{\alpha}{1-\gamma}\left[(z+\theta\eta-\lambda)^{1-\gamma}-(z-\lambda)^{1-\gamma}\right]-\gamma(z-\lambda)^{-\gamma}\theta\eta\right]dq(\eta)\\ &\quad=\int_{-1}^{\infty}\alpha[[(z+\theta\eta\upsilon-\lambda)^{-\gamma}-(z-\lambda)^{-\gamma}]\theta\eta]\,dq(\eta)\le 0\end{aligned}$$

for some $\upsilon\in(0,1)$. Therefore, for any $\alpha>0$, there exist constants $\xi>0$, and $\kappa\in(0,\lambda+\xi]$, such that $F(x,\psi,D\psi,D^2\psi)\le 0$, for every $x\in I_\xi\cup\Gamma_\kappa$. Define the stopping-time $\tau_0=\inf\{t\ge 0: Y_t\notin I_\xi\cup\Gamma_\kappa\}$. Then, using Itô's lemma

$$E[e^{-\rho\tau}\psi(Y_{\tau_0})]=\psi(y)+E\left[\int_0^{\tau_0}e^{-\rho t}(A^{c,\theta}\psi(Y_t)-\rho\psi(Y_t))\,dt+\sum_{0\le t\le\tau_0}(\psi(Y_t)-\psi(Y_{t^-}))\right]$$

where the last term represents the jumps caused by the durable good trading. If the agent trades the durable good, then $X_t=X_{t^-}-\lambda P_tK_t$ and $\psi(Y_t)\le\psi(Y_{t^-})$ for every $t\ge 0$. Therefore

$$E[e^{-\rho\tau}\psi(Y_{\tau_0})]\le\psi(y)-E\left[\int_0^{\tau_0}e^{-\rho t}U(C_t,K_t)\right]$$

If we chose $\alpha\ge\alpha_V((\lambda+\xi)/\xi)^{1-\gamma}$, then

$$\begin{aligned}\psi(X_{\tau_0},K_{\tau_0},P_{\tau_0})&=\frac{\alpha P_{\tau_0}^{-(1-\beta)(1-\gamma)}}{1-\gamma}(X_{\tau_0}-\lambda K_{\tau_0}P_{\tau_0})^{1-\gamma}\\ &\ge\frac{\alpha_V P_{\tau_0}^{-(1-\beta)(1-\gamma)}}{1-\gamma}X_{\tau_0}^{1-\gamma}\ge V(X_{\tau_0},K_{\tau_0},P_{\tau_0})\end{aligned}$$

where the last inequality follows from Theorem 3, item 1. □

A.3. Proof of Theorem 5

We assume that

$$\sup_{y\in[\lambda,\infty)}\{\underline{v}(y)-\overline{v}(y)\}>0 \tag{A.8}$$

and aim to find a contradiction. Using the sublinear growth of $\underline{v}(y)$, we can find a $\kappa>0$, $\omega>0$, such that $\underline{v}(y)\le\kappa(1+y)^{\omega}$ for all $y\in[\lambda,\infty)$. Let $\widehat{\omega}\in(\omega,1)$, then, by (A.8), for sufficiently small $\varepsilon>0$

$$\sup_{y\in[\lambda,\infty)}\{\underline{v}(y)-\overline{v}(y)-\varepsilon y^{\widehat{\omega}}\}>0 \tag{A.9}$$

Using the fact that $\underline{v}(y)$ and $\overline{v}(y)$ are continuous and increasing, that $\underline{v}(\lambda)=\overline{v}(\lambda)=0$, and that $\lim_{y\to\infty}\{\underline{v}(y)-\overline{v}(y)-\varepsilon y^{\widehat{\omega}}\}$, then the supremum in (A.9) is attained for a $y^*\in(\lambda,\infty)$

$$\sup_{y\in[\lambda,\infty)}\{\underline{v}(y)-\overline{v}(y)-\varepsilon y^{\widehat{\omega}}\}=\underline{v}(y^*)-\overline{v}(y^*)-\varepsilon {y^*}^{\widehat{\omega}}$$

Define the functions ψ, $\varphi:[\lambda,\infty)\times[\lambda,\infty)\to\mathbb{R}$ with $\varphi(y,z)=(\alpha(z-y)-4\phi)^4+\varepsilon z^{\widehat{\omega}}$ for $\alpha>1$, $\varepsilon>0$, $\phi>0$, and $\psi(y,z)=\underline{v}(z)-\overline{v}(y)-\phi(y,z)$. In order to continue our proof we shall need the following:

Claim 9. *The function $\psi(y,z)$ is bounded on $[\lambda,\infty)\times[\lambda,\infty)$ and attains its supremum in the compact set $[\lambda,\overline{\kappa})\times[\lambda,\overline{\kappa})$, where $\overline{\kappa}$ is a constant independent of α, ϕ and ε. The maximum point (y_α,z_α) has the following properties*:

1. $\lim_{\alpha\to\infty}|z_\alpha-y_\alpha|=0$
2. $\lim_{\alpha\to\infty}(\alpha(z-y)-4\phi)^4=0$
3. $\lim_{\varepsilon\to 0,\alpha\to\infty}\varepsilon z_\alpha^{\widehat{\omega}}=0$.

Claim 10 (Damgaard et al., 2003). *For all $\alpha>1$, $\phi>0$, and $\varepsilon>0$ there exist numbers $Z<0$ and $Y>0$, with $Y+Z>0$ such that*

$$\widehat{\rho}\underline{v}(z_a)\le G\left(z_a,\underline{v},\frac{\partial\psi}{\partial z},Z\right) \tag{A.10}$$

$$\widehat{\rho}\overline{v}(y_a)\le G\left(y_a,\overline{v},-\frac{\partial\psi}{\partial y},-Y\right) \tag{A.11}$$

and

$$\begin{pmatrix} Z & 0\\ 0 & y\end{pmatrix}\le\begin{pmatrix}\frac{\partial^2\psi}{\partial z^2} & \frac{\partial^2\psi}{\partial z\,\partial y}\\ \frac{\partial^2\psi}{\partial z\,\partial y} & \frac{\partial^2\psi}{\partial y^2}\end{pmatrix}=\begin{pmatrix}A+\varepsilon B & -A\\ -A & A\end{pmatrix}$$

with $A=12\alpha^2(\alpha(z_a-y_\alpha)-4\phi)^2\ge 0$ and $B=\widehat{\omega}(\widehat{\omega}-1)z_\alpha^{\widehat{\omega}}-2\le 0$, where $\widehat{\rho}=\rho-(1-\gamma)\left(\beta\mu_P-\frac{1}{2}\beta[1-\beta(1-\gamma)]\left(\sigma_{P_1}^2+\sigma_{P_2}^2\right)-\delta\right)$, and

$$G(z,v,a,b)=\sup_{\theta,c\,\in\,A(z)}\left\{\begin{array}{c} a\left[\begin{array}{c}(z-1)(r-\mu_P+\delta+(1-\beta(1-\gamma))(\sigma_{P_1}^2+\sigma_{P_2}^2))\\ -c+\theta(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P1})\end{array}\right]\\ +\frac{1}{2}b\left[(\theta\sigma-(z-1)\sigma_{P_1})^2+(z-1)^2\sigma_{P_2}^2\right]+\\ \int_{-1}^{L}[v(z+\theta\eta)-v(z)-a\theta\eta]\,dq(\eta)+\frac{c^{\beta(1-\gamma)}}{1-\gamma}\end{array}\right\}$$

Claim 11. $\lim_{\varepsilon\to 0}\lim_{\phi\to 0}\lim_{\alpha\to\infty}(\partial\psi/\partial z+\partial\psi/\partial y)=0$ *and* $\lim_{\varepsilon\to 0}\lim_{\phi\to 0}\lim_{\alpha\to\infty}(\widehat{c}^{\beta(1-\gamma)}-\tilde{c}^{\beta(1-\gamma)})=0$ *where $\widehat{c}$ ($\tilde{c}$) is the consumption control that maximizes $G(z_a,\underline{v},\partial\psi/\partial z,Z)$ ($G(y_\alpha,\overline{v},-\partial\psi/\partial y,-Y)$).*

Claim 12. $\underline{v}(z_\alpha+\widehat{\theta}\eta)-\underline{v}(z_\alpha)-(\partial\psi/\partial z)\widehat{\theta}\eta-(\overline{v}(y_\alpha+\tilde{\theta}\eta)-\overline{v}(y_\alpha)+(\partial\psi/\partial y)\tilde{\theta}\eta)\le 0$ *where*

$$\widehat{\theta}=\lim_{\varepsilon\to 0}\lim_{\phi\to 0}\lim_{\alpha\to\infty}\arg\max_{\theta}\left[\begin{array}{c}\frac{\partial\psi}{\partial z}\theta\left(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P_1}\right)+\frac{1}{2}Z\left(\theta^2\sigma^2-2(z_\alpha-1)\theta\sigma\sigma_{P_1}\right)\\ +\int_{-1}^{L}\left[\underline{v}(z_\alpha+\theta\eta)-\frac{\partial\psi}{\partial z}\theta\eta\right]\,dq(\eta)\end{array}\right]$$

and

$$\tilde{\theta}=\lim_{\varepsilon\to 0}\lim_{\phi\to 0}\lim_{\alpha\to\infty}\arg\max_{\theta}\left[\begin{array}{c}-\frac{\partial\psi}{\partial y}\theta\left(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P_1}\right)-\frac{1}{2}Y\left(\theta^2\sigma^2-2(z_\alpha-1)\theta\sigma\sigma_{P_1}\right)\\ +\int_{-1}^{L}\left[\overline{v}(z_\alpha+\theta\eta)+\frac{\partial\psi}{\partial y}\theta\eta\right]\,dq(\eta)\end{array}\right].$$

Using Eqs. (A.10) and (A.11) we have

$$\begin{aligned}\widehat{\rho}\left(\underline{v}(z_\alpha)-\overline{v}(y_\alpha)\right)\le{}&\sup_{\theta,c\,\in\,A(z_\alpha)}\left\{\begin{array}{c}\frac{\partial\psi}{\partial z}\left[\begin{array}{c}(z_\alpha-1)\begin{pmatrix}r-\mu_P+\delta+(1-\beta(1-\gamma))\\ \times(\sigma_{P_1}^2+\sigma_{P_2}^2)\end{pmatrix}\\ -c+\theta(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P_1})\end{array}\right]\\ +\frac{1}{2}Z\left[\left(\theta\sigma-(z_\alpha-1)\sigma_{P_1}\right)+(z_\alpha-1)^2\sigma_{P_2}^2\right]\\ +\int_{-1}^{L}\left[\underline{v}(z_\alpha+\theta\eta)-\underline{v}(z_\alpha)-\frac{\partial\psi}{\partial z}\theta\eta\right]\,dq(\eta)+\frac{c^{\beta(1-\gamma)}}{1-\gamma}\end{array}\right\}\\ &-\sup_{\theta,c\,\in\,A(y_\alpha)}\left\{\begin{array}{c}-\frac{\partial\psi}{\partial y}\left[\begin{array}{c}(y_\alpha-1)\begin{pmatrix}r-\mu_P+\delta+(1-\beta(1-\gamma))\\ \times(\sigma_{P_1}^2+\sigma_{P_2}^2)\end{pmatrix}\\ -c+\theta(\mu-r-(1-\beta(1-\gamma))\sigma\sigma_{P_1})\end{array}\right]\\ -\frac{1}{2}Y\left[\left(\theta\sigma-(y_\alpha-1)\sigma_{P_1}\right)+(y_\alpha-1)^2\sigma_{P_2}^2\right]\\ +\int_{-1}^{L}\left[\overline{v}(y_\alpha+\theta\eta)-\overline{v}(y_\alpha)-\frac{\partial\psi}{\partial z}\theta\eta\right]\,dq(\eta)+\frac{c^{\beta(1-\gamma)}}{1-\gamma}\end{array}\right\}\end{aligned}$$

Applying the claims above we get $\lim_{\varepsilon\to 0}\lim_{\phi\to 0}\lim_{\alpha\to\infty}\widehat{\rho}(\underline{v}(z_\alpha)-\overline{v}(y_\alpha))\le 0$, which contradicts the initial assumption.

The proofs of the first three claims follow closely Benth et al. (2002) and Damgaard et al. (2003). Regarding the last claim, note that

$$\begin{aligned}&\underline{v}\left(z_\alpha+\widehat{\theta}\eta\right)-\underline{v}(z_\alpha)-\frac{\partial\psi}{\partial z}\widehat{\theta}\eta-\left(\overline{v}\left(y_\alpha+\tilde{\theta}\eta\right)-\overline{v}\left(y_\alpha\right)+\frac{\partial\psi}{\partial y}\tilde{\theta}\eta\right)\\&\quad=\psi\left(z_\alpha+\widehat{\theta}\eta,y_\alpha+\tilde{\theta}\eta\right)-\psi\left(z_\alpha,y_\alpha\right)+\varepsilon(z_\alpha+\widehat{\theta}\eta)^{\widehat{\omega}}-\varepsilon(z_\alpha)^{\widehat{\omega}}-\frac{\partial\psi}{\partial z}\widehat{\theta}\eta-\frac{\partial\psi}{\partial y}\tilde{\theta}\eta\end{aligned}$$

and the limit of the last expression as $\varepsilon\to 0$, $\phi\to 0$, and $\alpha\to\infty$, is non-positive due to the claims above and to the fact that $\psi(z_\alpha+\widehat{\theta}\eta,y_\alpha+\tilde{\theta}\eta)\le\psi(z_\alpha,y_\alpha)$.

Appendix B

Our objective is to find a solution to problem (5)–(6). The solution $v(z)$ must satisfy $v(z)\ge((z-\lambda)^{1-\gamma}/(1-\gamma))M$, for every z belonging to the solvency region, where M is given in Eq. (4). Let us define $\underline{z}=\inf\{z\in A(z): v(z)>((z-\lambda)^{1-\gamma}/(1-\gamma))M\}$ and $\overline{z}=\sup\{z\in A(z): v(z)>((z-\lambda)^{1-\gamma}/(1-\gamma))M\}$, then we impose the following value matching and smooth pasting conditions:

$$v(\underline{z})=\frac{(\underline{z}-\lambda)^{1-\gamma}}{1-\gamma}M \tag{B.1}$$

$$v(\overline{z})=\frac{(\overline{z}-\lambda)^{1-\gamma}}{1-\gamma}M \tag{B.2}$$

$$v'_{+}(\underline{z})=(\underline{z}-\lambda)^{-\gamma}M \tag{B.3}$$

$$v'_{-}(\overline{z})=(\underline{z}-\lambda)^{-\gamma}M \tag{B.4}$$

where v'_{-}, and v'_{+} denote the left and right derivatives, respectively. A solution to this problem is given by the value function $v(z)$, and the values $(\underline{z},\overline{z},M)$, such that (5)–(6) and the conditions above are satisfied.

We solve this problem using and adaptation of the algorithm used in Damgaard et al. (2003) and Grossman and Laroque (1990).

Step 1: Guess $\underline{z}$.

Step 2: Guess $M=M_0$, and solve the problem (5)–(6) through value function iteration[8] on the interval $(\underline{z},z_{\max})$. Define $\overline{z}=\inf\{z>\underline{z}: v(z)=((z-\lambda)^{1-\gamma}/(1-\gamma))M\}$ and make $v(z)=((z-\lambda)^{1-\gamma}/(1-\gamma))M$ for $z\in[\overline{z},z_{\max}]$. Iterate until convergence and calculate the new M using Eq. (4). If the new $M=M_0$, proceed to step 3. Otherwise, guess a new M_0 and repeat step 2.

Step 3: Verify if the smooth pasting condition (B.4) is satisfied. If (B.4) holds we have a solution. Otherwise, return to step 1.

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[8] See Kushner and Dupuis (1992), for example.

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