American Derivatives in Dry Markets

João Amaro de Matos
Faculdade de Economia
Universidade Nova de Lisboa
Campus de Campolide
1099-032 Lisboa, Portugal

Ana Lacerda
Banco de Portugal

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Abstract

This paper studies the impact of dry markets for underlying assets on the pricing and optimal exercise of American derivatives. Dry markets are characterized by the possibility of non-existence of trading at certain dates. Such non-existence may be deterministic or probabilistic. Using superreplicating strategies, we derive expectation representations for the range of arbitrage-free values of the derivatives. In the probabilistic case, if we consider an enlarged filtration induced by the price process and the market existence process, ordinary stopping times are required. If not, randomized stopping times are required. Several comparisons of the ranges obtained with the two market restrictions are performed. Finally, we conclude that market incompleteness delays the optimal exercise of American derivatives, although there may exist moments when there is indifference between exercising and selling the American derivative

1 Introduction

Among the traditional assumptions on which derivatives' pricing is based, markets are perfect and the underlying asset can be transacted at any point in time. Under the absence of arbitrage opportunities the value of a derivative can be computed as the value of a portfolio on the underlying risky asset and risk-free bonds that exactly replicates its payoff. Such portfolio can be rebalanced in a self-financing way until the maturity of the derivative, by continuously transacting the underlying asset and the bonds. Under these assumptions, the calculated value of the initial portfolio can be shown to be the equilibrium price of the derivative. Considering the case of American derivatives it has been shown by Bensoussan (1984) and Karatzas (1988) that, in this setting, the no-arbitrage value of one such derivative is indeed the supremum of the implied European derivative values over all possible stopping times.

In this paper we assume that an American derivative and its respective underlying asset may not be transacted at some points in time, generating incomplete markets, and study the impact of this constraint on the pricing of American derivatives. In particular, we are able to characterize some features of the optimal exercise policy of such derivatives, and to write the upper and lower bounds for their possible equilibrium values in terms of both randomized and ordinary stopping times.

The fact that the assets can be transacted only at some points in time can be described as a lack of liquidity of the market, as in Longstaff (2001). We shall refer to this situation as *dry markets*. We will consider two different types of dry markets. In the first type, to be called the *deterministic case*, we know ex-ante exactly at which points in time markets do exist or do not exist. In the second type, to be called the *probabilistic case*, we assign a probability p to the existence of the market at each point in time.

Markets' dryness implies that markets may become incomplete in the sense that perfect hedging of the derivative in all states of nature is no longer possible. However, for any given derivative, portfolios can be found that have the same payoff as the derivative in some states of nature and higher payoffs in the other states. Such portfolios are said to be *superreplicating* (or superhedging). Holding one such portfolio should be worth more than the derivative itself and therefore, the value of the cheapest of such portfolios should be seen as a bound on the value of the derivative. The nature of the superreplicating bounds for European derivatives is well characterized in the context of incomplete markets in the papers by El Karoui and Quenez (1991,1995), Edirisinghe, Naik and Uppal (1993) and Karatzas and Kou (1996). A direct application to the case of European option pricing when the market for the underlying is dry can be found in Amaro de Matos and Antão (2001). As all these results stress, under market incompleteness the hedging position of a market-maker is different depending on whether this intermediary is in a long or in a short position. This fact results in a lower and an upper bound for the derivatives' values.

The superreplicating bounds establish the limits of the interval for the prices outside which an investor has a positive profit with probability one. In other words, an arbitrage opportunity exists if the investor sells options above the upper bound or buys options below the lower bound.

There has been a relatively extensive literature in the continuous time setting, analyzing this problem and characterizing in varying degrees of generality the superhedging bounds of American derivatives in incomplete markets. Examples are the papers by Kramkov (1996), Follmer and Kramkov (1997), Follmer and Kabanov (1998) and Karatzas and Kou (1998). More recently, a paper by Chalasani and Jha (2001) discusses the particular case of transaction costs in discrete time and conclude that, in their specific setting, the superreplicating bounds of one such derivative may also be written as the supremum of the implied European derivative value. However, there are two important subtleties in their result: first, the supremum in this case must be taken over *randomized* stopping times and second, the probability measure defining the European value over which the supremum is taken, may depend itself on the randomized stopping time that solves the problem.

Chalasani and Jha (2001) relate their result to the fact that[1], under

[1] For a discussion of this point see, among others, Duffie (2001), p.37.

incomplete markets, the choice of exercise policy may influence the characterization of the marketed subspace, and therefore influence the pricing of securities. A rational exercise policy may even not be well defined if the state-price deflator depends on the exercise policy. This argument would provide solid ground for the optimal randomized stopping times characterizing the superreplication bounds of the American derivatives under proportional transaction costs.

Our results show that, under dry markets, and in the same general discrete time setting used by Chalasani and Jha (2001), we can also write the superreplicating bounds of an American derivative as the supremum of the implied European derivative value. However, the supremum in this case may be taken over *deterministic* stopping times, as opposed to the intuition provided by the above cited authors. Although the result for deterministic dry markets may be understood in the context of the superreplicating bounds discussed in Harrison and Kreps (1979), the case of probabilistic dry markets is of a different nature since it crosses an additional source of uncertainty (existence or non existence of the market at a given point in time). Furthermore, we show that market incompleteness may delay the optimal exercise of American derivatives.

Our work is organized as follows. Section 2 models the deterministic case, introducing the model and relevant probabilistic concepts. Section 3 states the corresponding results, presenting the upper and lower superreplicating bounds of American derivatives. This is followed by Section 4 that models the probabilistic case, after what Section 5 presents the corresponding results for the upper and lower superreplicating bounds of American derivatives. In section 6 these different bounds are compared. The exercise policy in dry markets is discussed in section 7. Finally, we section 8 we conclude. Our main technical proofs are presented in the Appendix.

2 Deterministic Dry Markets

2.1 The model

Consider an economy where three different assets are transacted. The first asset is a risk free asset with unitary initial value that provides a certain total return of R per period; the second asset to be considered is a risky asset (the stock); finally, the third asset is an American derivative, written on the stock, with expiration date T. We work in discrete time, corresponding to dates $0, 1, ..., T$. The set of these dates is denoted by $\mathcal{T} \equiv \{0, 1, \ldots, T\}$. The evolution of the value of the underlying asset is modelled by means of a finite *event tree*. Each node of such tree is identified by a pair (j, t), where j denotes the j-*th* node at time t. There is only one node at time $t = 0$, denoted by $(0, 0)$. For any given node (j, t), the set of successors at time $t + k$, $k > 0$, is denoted by $j_t^+(t + k)$. For simplicity let j_t^+ denote the set of immediate successors, i.e., $j_t^+ \equiv j_t^+(t + 1)$. The nodes (j, T), at time T, are called terminal nodes and j_T^+ is assumed to be the empty set $\emptyset$. It is also assumed that, for $t < T$, each nonterminal node (j, t) has a nonempty set of immediate successors, i.e., $j_t^+ \neq \emptyset$. In an analogous way, the set of immediate predecessors of a node $(j, t) \neq (0, 0)$ is denoted by j_t^-. In what follows we shall consider the case where such sets j_t^- have a unique element. Moreover, we denote by $\mathcal{J}_t$ the set of all nodes at any point in time t

$$\mathcal{J}_t = \cup_j (j, t).$$

A *path* on the event tree is a set of nodes $w = \cup_{t \in \{0,1,\ldots,T\}} (j_t, t)$ such that each element in the union satisfies $(j_{t+k}, t + k) \in j_t^+ (t + k)$, with $k > 0$ and $t + k \in \{0, 1, ..., T\}$. Let Ω denote the set of all paths on the event tree. Each node in the tree represents the set of all tree paths that contain that node. Let S denote the process followed by the stock price. More precisely, let $S(j, t)$ denote the price of the stock at node (j, t). A natural *filtration* on the space Ω associated to the price process S is $\mathcal{F} = \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T$, where each $\mathcal{F}_t$ is the σ-algebra generated by the random variable $S(\cdot, t)$. All the random variable will be defined in the measurable space $(\Omega, \mathcal{F})$. Similarly, let G denote the process followed by the payoff of American derivative. Hence, $G(j,t)$ denotes the payoff of the American derivative at node (j,t) whenever exercised at that point. Let $\bar{S}(j,t)$ and $\bar{G}(j,t)$ stand for the discounted values of the above processes, *i.e.*,

$$\bar{G}(j,t) = \frac{G(j,t)}{R^t} \text{ and } \bar{S}(j,t) = \frac{S(j,t)}{R^t}.$$

Dry markets are characterized by the fact that transactions are possible only at some points in time. We hereby model dry markets allowing transactions only at times t in a set $\mathcal{T}_m \subseteq \mathcal{T}$. It is also assumed that transactions are possible at times $t = 0$ and $t = T$, *i.e.*, $\{0, T\} \subseteq \mathcal{T}_m$.

At any node (j,t) consider the portfolio constituted by $\Delta(j,t)$ shares of the underlying asset and an amount $B(j,t)$ invested in the risk free asset. One such portfolio is denoted by $[\Delta(j,t), B(j,t)]$. Its value process is given by

$$V(j,t) = \Delta(j,t) S(j,t) + B(j,t).$$

Consider a short position on the American derivative. A *replicating strategy* is a sequence of portfolios $\{[\Delta(j,t), B(j,t)]\}_{t \in \mathcal{T}_m}$ such that the value of each of them is larger than or equal to the payoff of the derivative at any non-terminal node in the next transaction time. Additionally, at any terminal node its value is equal to the payoff of the derivative. In other words, for any two consecutive trading dates t_1 and $t_2 > t_1$, consider an arbitrary node (j, t_1) and the subset of its possible successors $j_{t_1}^{+}(t_2)$. Then, the portfolio at t_1, $[\Delta(j,t_1), B(j,t_1)]$, must be such as to generate in t_2 a value $\Delta(j,t_1) S(i,t_2) + B(j,t_1) R^{t_2 - t_1}$ such that

$$\Delta(j,t_1) S(i,t_2) + B(j,t_1) R^{t_2 - t_1} \geq G(i,t_2)$$

with $(i,t_2) \in j_{t_1}^{+}(t_2)$ and if $t_2 = T$ then

$$\Delta(j,t_1) S(i,T) + B(j,t_1) R^{T - t_1} = G(i,T).$$

A *self-financed portfolio* is a portfolio that generates enough wealth to rebalance the portfolio according to any future state of nature. In other words, for any two consecutive trading dates t_1 and $t_2 > t_1$, consider an arbitrary node (j, t_1) and the set of its possible successors $\left\{(i, t_2) : i \in j_{t_1}^{+}\right\}$. Then, the value of the portfolio at that point in time, $\Delta(j, t_1) S(j, t_1) + B(j, t_1)$ must be such as to generate

$$\Delta(j, t_1) S(i, t_2) + B(j, t_1) R^{t_2 - t_1} \geq V(i, t_2).$$

For a long position on the American derivative, analogous definitions are obtained with reverted inequalities. Moreover, if a long position in the derivative is considered, superreplication only applies at the nodes (i, t) where the option is be exercised.

If a complete market is considered the value of an American derivative is the value of the cheapest self-financing portfolio on the underlying risky asset and risk-free bonds that replicates the payoff of the American derivative. For this portfolio the following condition will hold at any non-terminal node

$$\Delta(j, t_1) S(i, t_2) + B(j, t_1) R^{t_2 - t_1} = \max\left[V(i, t_2), G(i, t_2)\right].$$

In dry markets, however, the number of transacted securities may be insufficient to allow the construction of a self-financing portfolio that replicates the payoff of an American derivative. In other words, markets may become incomplete. In that case, there is not a unique arbitrage free value for the American derivative. However, replacing the notion of replicating strategy by the notion of *superreplication strategy* it is possible to derive an arbitrage free range of variation for the value of the American derivative. In order to find the upper bound of this range consider a short position in the derivative. The upper bound will be the value of the cheapest portfolio that the buyer of the derivative can buy in order to completely hedge against any possibility of exercise of the American derivative and without need of additional financing at any rebalancing dates. Note that in order to completely hedge against the possibility of exercise the value of the portfolio, at any given node, has to be equal or higher than the payoff of the American derivative. In that case it is said that the portfolio *superreplicates* the payoff of the American derivative. On the other hand, in order to find the lower bound of the arbitrage-free range of variation consider a long position in the derivative. For a given exercise policy consider the most expensive portfolio that the buyer of the American derivative can buy in order to be fully hedged. The lower bound is the value of the most expensive portfolio chosen among the portfolio just described. Note that in this case the buyer of the American derivative is, for a given exercise policy, completely hedged if in any node where the option may be exercised the payoff of the American derivative is higher than the value of the hedging portfolio. In this case it is said that the portfolio is *superreplicated* by the American derivative.

Under market completeness, both limiting portfolios coincide with a replicating portfolio and the value of the derivative is well characterized [Karatzas (1988)]. Under market incompleteness however, that is no longer true and the arbitrage-free value of the derivative must lie between the values of the two limiting superreplicating portfolios.

In what follows we are going to characterize the upper and lower arbitrage-free bounds for the value of the American derivatives in the framework described above. In order to do that, we first define some mathematical objects, such as node probability measure, adjusted probability measure and stopping time.

2.2 Some Probabilistic Definitions

Definition 2.1 *A **node probability measure** is a nonnegative node function $q(i,t)$ satisfying*

$$\sum_{t\in\mathcal{T}_m}\sum_{(i,t)\in\mathcal{J}_t} q(i,t) = 1.$$

The set of all node probability measures is denoted by Q.

Definition 2.2 *A node probability measure on the event tree is said to be **simple** if, for $t \in \mathcal{T}_m$ and $t+k \in \mathcal{T}_m$, there are no two nodes in the same path, say (i,t) and $(j,t+k) \in i_t^+(t+k)$, such that $q(i,t) > 0$ and $q(j,t+k) > 0$.*

The following theorem is analogous to theorem 6.7 of Chalasani and Jha (2001) but now in the framework of dry markets.

Definition 2.3 *A node probability measure $q \in Q$ is said to be a **node martingale measure** if, for all $(i,t) \in \mathcal{J}_t$ and $t \in \mathcal{T}_m$, satisfies*

$$\sum_{\substack{\tau>t \\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}_t^+(\tau)} q(j,\tau)\,\bar{S}(i,t) = \sum_{\substack{\tau>t \\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}_t^+(\tau)} q(j,\tau)\,\bar{S}(j,\tau).$$

The set of all node martingale measures is denoted by $Q^{\mathcal{M}}$.

Theorem 2.1 (Chalasani and Jha) *The extreme points of the set $Q^{\mathcal{M}}$ are simple node probability measures, i.e., on every path on the event tree there is at most one node where q is strictly positive.*

The proof of this theorem follows closely the proof of theorem 6.7 in Chalasani and Jha (2001) and is presented in the appendix A.

Definition 2.4 *An **adjusted probability measure** is a nonnegative function $P(i,t)$ such that $P(0,0)=1$ and for all $t \in \mathcal{T}_m$*

$$P(i,t) = \sum_{(j,s)\in i^+(s)} P(j,s)$$

with $s = \min\{z \in \mathcal{T}_m : z > t\}$.

The set of all probability measures is denoted by $\mathbb{P}$.

Definition 2.5 *A process $Z = \{Z_t : t \in \mathcal{T}_m\}$ is called **adapted** to the filtration $\mathcal{F}$ if for each $t \in \mathcal{T}_m$, Z_t is $\mathcal{F}_t$-measurable.*

Let τ denote an ordinary stopping time that takes values in $\mathcal{T}_m$, *i.e.*, τ is a map such that $\tau : \Omega \to \mathcal{T}_m$ and $\{w : \tau(w) \leq t\} \in \mathcal{F}_t$ for all $t \in \mathcal{T}_m$. We define a nonnegative adapted process X_τ associated with τ that is defined for all $t \in \mathcal{T}_m$ and has the form $X_\tau(i,k) = 1$ if $\tau(w) = k$ and $X_\tau(i,k) = 0$ otherwise, where (i,k) is a node in path w. Let $\mathbb{T}$ and $\mathbb{X}_{\mathbb{T}}$ denote the set of all τ and associated X_τ, respectively.

Definition 2.6 *A simple node probability measure is said to be associated with a given stopping time if at any node such that $X_\tau(i,k)$ is equal to zero then $q(i,t)$ is also equal to zero. Moreover, at any node such that $X_\tau(i,k)$ is strictly positive then $q(i,t)$ is also positive.*

The set of all node probability measures with this property is denoted by Q^τ.

Definition 2.7 *For any adjusted probability measure $P \in \mathbb{P}$ and stopping time $\tau \in \mathbb{T}$ we say that P is a τ-**martingale measure** if, P-almost surely, for any (i,t) with $t \in \mathcal{T}_m$ we have*

$$\sum_{m>t, m\in\mathcal{T}_m} \sum_{(j,m)\in i_t^+(m)} p(j,m) X_\tau(j,m) \left[\bar{S}(i,t) - \bar{S}(j,m)\right] = 0,$$

The set of all P that have this property is denoted by $\mathcal{P}(\tau)$.

Definition 2.8 *For any adjusted probability measure $P \in \mathbb{P}$ we say that P is a **martingale measure** if, P-almost surely, for any $(i,t) \in \mathcal{J}_t$ with $t \in \mathcal{T}_m$ we have*

$$\sum_{(j,m)\in i_t^+(m)} p(j,m) \left[\bar{S}(i,t) - \bar{S}(j,m)\right] = 0,$$

where $m \in \mathcal{T}_m$.

The set of all P that have this property is denoted by $\mathcal{P}$.

Let (P, X_τ) denote a measure-strategy pair, *i.e.*, a pair constituted by an adjusted probability measure and a nonnegative adapted process.

Definition 2.9 *A measure-strategy pair* (P, X_τ) *is said to be* ***equivalent*** *to a node probability measure if* $P(i,t)\, X_\tau(i,t) = q(i,t)$ *for any given node* (i,t)

We can now enunciate a fundamental result.

Theorem 2.2 (Chalasani and Jha) *Let* (P, X_τ) *be a measure-strategy pair. The simple node function* q *defined by* $q(i,t) = P(i,t)\, X_\tau(i,t)$ *is the unique equivalent node-measure. Conversely, for a given simple node probability measure* q*, there is a measure-strategy pair* (P, X_τ) *equivalent to* q*, such that* P *and* X_τ *are uniquely defined at nodes* (i,t) *where* $q(i,t) + \sum_{\substack{(j,\tau)\in i_t^+(\boldsymbol{\tau}) \\ \tau>t,\ \tau\in\mathcal{T}_m}} q(j,\tau) > 0$.

A version of the proof of this result, adjusted to case of dry markets, is provided in Appendix A.

3 Results on Deterministic Dry Markets

3.1 Upper bound for the Value of an American Derivative

The upper bound for the value of an American derivative is the maximum value for which the derivative would be transacted without allowing for arbitrage opportunities. In order to find the upper bound consider a short position in the derivative. The maximum value for which the derivative would be transacted without allowing for arbitrage opportunities would be the value of the cheapest portfolio that the buyer of the derivative can buy in order to completely hedge against any possibility of exercise of the American derivative and without need of additional financing at any rebalancing dates. A portfolio is initially built such that, at each transaction date until maturity, it generates enough wealth, so as to be rebalanced according to any revealed state of nature. Since by construction there is no need of additional financing, one such strategy is said to be a *self-financed strategy.*

Additionally, it has to be a *superreplicating strategy, i.e.,* a sequence of portfolios $\{[\Delta(j,t), B(j,t)]\}_{t\in\mathcal{T}_m}$ such that the value of each of them is larger than or equal to the payoff of the derivative at any non-terminal node in the next transaction time. In other words, for any two consecutive trading dates t_1 and $t_2 > t_1$, consider an arbitrary node (j,t_1) and the subset of its possible successors $j^+_{t_1}(t_2)$. Then, the portfolio at t_1, $[\Delta(j,t_1), B(j,t_1)]$, must be such as to generate in t_2 a value $\Delta(j,t_1) S(i,t_2) + B(j,t_1) R^{t_2-t_1}$ such that

$$\Delta(j,t_1) S(i,t_2) + B(j,t_1) R^{t_2-t_1} \geq G(i,t_2), \tag{1}$$

with $(i,t_2) \in j^+_{t_1}(t_2)$.

More formally, take any $t_1 \in \mathcal{T}_m$, such that $t_1 \neq T$. Define the consecutive trading date as $t_2 = \min(s \in \mathcal{T}_m : s > t_1)$. The upper bound for the value of the American derivative can thus be seen as the solution of the following problem:

$$V^u_d = \min_{\substack{\{\Delta(j,t), B(j,t)\} \\ (j,t)\in\mathcal{J}_t \\ t\in\mathcal{T}_m\setminus\{T\}}} \Delta(0,0) S(0,0) + B(0,0)$$

subject to the superreplicating constraints:

$$\Delta(0,0) S(0,0) + B(0,0) \geq G(0,0), \tag{2}$$

$$\Delta(j,t_1) S(i,t_2) + B(j,t_1) R^{t_2-t_1} \geq G(i,t_2), \tag{3}$$

for all $t_1 \in \mathcal{T}_m\setminus\{T\}$ and $(i,t_2) \in j^+(t_2)$ with $t_2 = \min(s \in \mathcal{T}_m : s > t_1)$ and subject to the self-financing constraints:

$$\Delta(j,t_1) S(i,t_2) + B(j,t_1) R^{t_2-t_1} \geq V(i,t_2) \tag{4}$$

for all $t_1 \in \mathcal{T}_m\setminus\{\max\{t\in\mathcal{T}_m : t < T\}, T\}$ and $(i,t_2) \in j^+(t_2)$ with $t_2 = \min(s \in \mathcal{T}_m\setminus\{T\} : s > t_1)$.

Using results from linear programming the upper bound of the range of variation for the arbitrage-free value of an American derivative can be written as follows.

Theorem 3.1 *There exists a node probability measure $q \in Q^{\mathcal{M}}$ such that the upper hedging price of an American derivative in a dry market can be written as*

$$V_d^u = \max_{q \in Q^{\mathcal{M}}} \sum_{t \in \mathcal{T}_m} \sum_{(j,t) \in \mathcal{J}_t} q\left(j,t\right) \bar{G}\left(j,t\right).$$

Proof. As the problem that must solved in order to find the upper bound of the American derivative is a linear programming problem it is possible to construct its dual. Let $\lambda\left(0,0\right), \lambda\left(i,t_2\right)$ and $\alpha\left(i,t_2\right)$ be the dual variables associated with constraints (2), (3) and (4), respectively. Then, the dual problem is

$$\max_{\lambda(j,t),\alpha(j,t)} \sum_{t \in \mathcal{T}_m} \sum_{(j,t) \in \mathcal{J}_t} \lambda\left(j,t\right) G\left(j,t\right)$$

subject to

$$\lambda\left(0,0\right) S\left(0,0\right) + \sum\nolimits_{(i,t) \in i_0^+(t)} \left[\lambda\left(i,t\right) + \alpha\left(i,t\right)\right] S\left(i,t\right) = S\left(0,0\right) \tag{5}$$

$$\lambda\left(0,0\right) + \sum\nolimits_{(i,t) \in i_0^+(t)} \left[\lambda\left(i,t\right) + \alpha\left(i,t\right)\right] R^t = 1 \tag{6}$$

with $t = \min\left(s \in \mathcal{T}_m : s > 0\right),$

$$\sum\nolimits_{(j,t_2) \in i_{t_1}^+(t_2)} S\left(j,t_2\right)\left[\lambda\left(j,t_2\right) + \alpha\left(j,t_2\right)\right] - \alpha\left(i,t_1\right) S\left(i,t_1\right) = 0 \tag{7}$$

$$\sum\nolimits_{(j,t_2) \in i_{t_1}^+(t_2)} \left[\lambda\left(j,t_2\right) + \alpha\left(j,t_2\right)\right] R^{t_2 - t_1} - \alpha\left(i,t_1\right) = 0 \tag{8}$$

for all $t_1 \in T \backslash \left\{0, \max\left\{s \in \mathcal{T}_m : s < T\right\}, T\right\}$ and $t_2 = \min\left(s \in \mathcal{T}_m : s > t_1\right),$ and, finally,

$$\sum\nolimits_{(j,T) \in i_t^+(T)} S\left(j,T\right) \lambda\left(j,T\right) - \alpha\left(i,t\right) S\left(i,t\right) = 0 \tag{9}$$

$$\sum\nolimits_{(j,T) \in i^+(T)} S\left(j,T\right) R^{T-t} - \alpha\left(i,t\right) = 0 \tag{10}$$

for all $t = \max\left\{s \in \mathcal{T}_m : s < T\right\}.$

Note that the constraints (5), (7) and (9) of the dual problem are associated with the variables $\Delta(0,0)$, $\Delta(i,t_1)$ and $\Delta(i,t)$, respectively, of the primal problem. In a similar way, the constraints (6), (8) and (10) are, respectively, associated with the primal variables $B(0,0)$, $B(i,t_1)$ and $B(i,t)$.

The constraints presented in equations (7), (8), (9) and (10) can be rewritten such that, for all $t \in \mathcal{T}_m \backslash T$, we have

$$\sum\nolimits_{\substack{(j,m)\in i_t^+(m) \\ m>t, m\in\mathcal{T}_m}} \lambda(j,m) R^{m-t} S(i,t) = \sum\nolimits_{\substack{(j,m)\in i_t^+(m) \\ m>t, m\in\mathcal{T}_m}} \lambda(j,m) S(j,m), \quad (11)$$

From equations (6), (8) and (10) we obtain

$$\sum\nolimits_{\substack{(i,t) \\ t\in\mathcal{T}_m}} \lambda(i,t) R^t = 1 \quad (12)$$

Considering equations (12), (5) and (12), we have, for all (i,t),

$$\sum\nolimits_{\substack{(j,m)\in i^+(m) \\ m>t, m\in\mathcal{T}_m}} \lambda(j,m) R^{m-t} S(i,t) = \sum\nolimits_{\substack{(j,m)\in i^+(m) \\ m>t, m\in\mathcal{T}_m}} \lambda(j,m) S(j,m).$$

Let $q(i,t) = \lambda(i,t) R^t$. For any $t \in \mathcal{T}_m \backslash \{T\}$,

$$\sum\nolimits_{\substack{(j,m)\in i^+(m) \\ m>t, m\in\mathcal{T}}} q(j,m) \bar{S}(i,t) = \sum\nolimits_{\substack{(j,m)\in i^+(m) \\ m>t, m\in\mathcal{T}}} q(j,m) \bar{S}(j,m).$$

Hence, the dual problem can be written as

$$\max_{q\in Q^{\mathcal{M}}} \sum\nolimits_{\substack{(i,t)\in\mathcal{J}_t \\ t\in\mathcal{T}_m}} q(i,t) \bar{G}(i,t).$$

■

The upper bound solving the problem above can also be seen as the solution of a more intuitive problem. In fact, it can be shown that this upper bound maximizes over all possible stopping times the expected discounted payoff, when the expectation is optimized among all adjusted probability measures. In other words,

Theorem 3.2 *There exists an adjusted probability measure $P \in \mathcal{P}(\tau)$ and an adapted process $X_\tau \in \mathbb{X}_\mathbb{T}$ such that the upper hedging price of an American derivative in a dry market can be written as*

$$V_d^u = \max_{\tau \in \mathbb{T}} \max_{P \in \mathcal{P}(\tau)} E^p G_\tau,$$

with $G_\tau(i,t) = X_\tau(i,t)\, G(i,t)$. Additionally, if there is a probability measure with positive probability on every path then the upper hedging price of an American derivative in a dry market can be rewritten as

$$V_d^u = \max_{\tau \in \mathbb{T}} \max_{P \in \mathcal{P}} E^p G_\tau,$$

where $G_\tau(i,t) = X_\tau(i,t)\, G(i,t)$, as before.

Proof. As Q is a convex set, the maximum of the problem

$$\max_{q \in Q^{\mathcal{M}}} \sum\nolimits_{\substack{(i,t) \in \mathcal{J}_t \\ t \in \mathcal{T}_m}} q(i,t)\, \bar{G}(i,t)$$

is obtained at the extremes points of $Q^{\mathcal{M}}$.

By theorem (2.1), we know that the extremes points are simple node measures. Using theorem (2.2) we can rewrite the problem above as

$$\max_{\tau \in \mathbb{T}} \max_{P \in \mathcal{P}(\tau)} E^p \bar{G}_\tau \tag{13}$$

where

$$\bar{G}_\tau(i,t) = \bar{G}(i,t)\, X_\tau(i,t).$$

As stressed in Chalasani and Jha (2001), page 64, if there is a martingale measure $\hat{P} \in \mathcal{P}$ with positive measure on every path, w, the inner maximization in (13) can be restricted to all $P \in \mathcal{P}$ without affecting its value. First, any $P \in \mathcal{P}$ also belongs to $\mathcal{P}(\tau)$. Second, any measure $P \in \mathcal{P}(\tau)$ can be redefined to be a martingale measure $P' \in \mathcal{P}$ such that $E^{P'} \bar{G}_\tau = E^P \bar{G}_\tau$, as follows

$$P'(i,t) = \begin{cases} P(i,t), & t \le k : if \;\; (i,t) \in w \text{ and } \tau(w) = k \\ P'_-(i,t)\, \frac{\hat{P}(i,t)}{\hat{P}_-(i,t)} & \text{otherwise} \end{cases}$$

where $P'_{-}(i,t)$ and $\hat{P}_{-}(i,t)$ stand for the probabilities in the node that is an immediate predecessor of (i,t). ■

3.2 Lower bound for the Value of an American Derivative

The lower bound for the value of an American derivative is the minimum value for which the derivative would be transacted without allowing for arbitrage opportunities. In order to find the lower bound consider a long position in the derivative. As stressed in Karatzas and Kou (1998) while the seller of the American derivative has to be hedged against any possible exercise policy, the buyer of the American derivative needs only to hedge against a given exercise policy that is defined by himself. For a given exercise policy consider the most expensive portfolio that the buyer of the American derivative can buy in order to be fully hedged and without need of additional financing at any rebalancing dates. The minimum value for which the derivative would be transacted without allowing for arbitrage opportunities would be the value of the most expensive portfolio chosen among all the portfolios just mentioned.

For any given stopping time τ and any node (j,t), such that (j,t) is before the exercise time, consider the portfolio constituted of $\Delta^{\tau}(j,t)$ shares of the underlying asset and an amount $B^{\tau}(j,t)$ invested in the risk free asset. For each stopping time we are looking for the most expensive portfolio that the buyer of the American derivative can buy that is *self-financed* and is *superreplicated* by the payoffs of the American derivative. A portfolio $[\Delta^{\tau}(j,t_1), B^{\tau}(j,t_1)]$ is said self-financing if, for any two consecutive trading dates t_1 and $t_2 > t_1$ and an arbitrary node (j,t_1), the portfolio is such as to generate in t_2 a value $\Delta^{\tau}(j,t_1)\, S(i,t_2) + B^{\tau}(j,t_1)\, R^{t_2-t_1}$ such that

$$\Delta^{\tau}(j,t_1)\, S(i,t_2) + B^{\tau}(j,t_1)\, R^{t_2-t_1} \leq V^{\tau}(i,t_2), \tag{14}$$

for any node $(i,t_2) \in j_{t_1}^{+}(t_2)$ before the exercise of the American derivative. Additionally, a portfolio $[\Delta^{\tau}(j,t_1), B^{\tau}(j,t_1)]$ is said to be superreplicated by $G_\tau(i,t_2)$ if, for any two consecutive trading dates t_1 and $t_2 > t_1$ and an arbitrary node (j,t_1), it generates in t_2 a value $\Delta^\tau(j,t_1)S(i,t_2)+B^\tau(j,t_1)R^{t_2-t_1}$ such that

$$\Delta^\tau(j,t_1)S(i,t_2)+B^\tau(j,t_1)R^{t_2-t_1}\le G_\tau(i,t_2), \tag{15}$$

for any $(i,t_2)\in j_{t_1}^+(t_2)$ when it is optimal to the holder of the American option to exercise it given τ. The minimum value for which the derivative would be transacted without allowing for arbitrage opportunities would be the value of the most expensive portfolio chosen among all stopping times.

The lower bound for the value of the American derivative can thus be seen as the solution of the following problem:

$$V_d^l=\max_{\tau\in\mathbb{T}}\max_{\{\Delta^\tau(j,t),B^\tau(j,t)\}}\Delta_\tau(0,0)S(0,0)+B_\tau(0,0)$$

subject to the superreplicating constraint

$$\Delta_\tau(0,0)S(0,0)+B_\tau(0,0)\le G_\tau(0,0),$$

if $\tau(w)=0$ and, otherwise,

$$\Delta_\tau(j,t_1)S(m,t_3)+B_\tau(j,t_1)R^{t_3-t_1}\le G_\tau(m,t_3),$$

for all (j,t_1) and (m,t_3) such that $X_\tau(m,t_3)=1$, $t_3=\min\{s\in\mathcal{T}_m : s>t_1\}$ and $(m,t_3)\in j_{t_1}^+(t_3)$; and to the self-financing constraints

$$\Delta_\tau(j,t_1)S(i,t_2)+B_\tau(j,t_1)R^{t_2-t_1}\le\Delta_\tau(i,t_2)S(i,t_2)+B_\tau(i,t_2),$$

for all (j,t_1) and $(i,t_2)\in j_{t_1}^+(t_2)$ such that $X_\tau(i,t_2)=0$, $t_3=\min\{s\in\mathcal{T}_m : s>t_1\}$ and for some (m,t_3), such that $X_\tau(m,t_3)=1,(m,t_3)\in j_{t_2}^+(t_3)$.

Using results from linear programming the upper bound arbitrage free bound of the American derivative can be written as follows.

Theorem 3.3 *There exists a node probability measure $q \in Q^{\tau}$ and a stopping time $\tau \in \mathbb{T}$ such that the lower hedging price of an American derivative in a dry market can be written as*

$$V_d^l = \max_{\tau \in \mathbb{T}} \min_{q \in Q^{\tau}} \sum\nolimits_{(j,t)} \sum\nolimits_{t \in \mathcal{T}_m} q(j,t)\, \bar{G}_{\tau}(j,t)$$

with $\bar{G}_{\tau}(j,t) = \bar{G}(j,t)\, X_{\tau}(j,t)$ and for any (i,t) and $t \in \mathcal{T}_m$

$$\sum_{m>t,\ m \in \mathcal{T}_m} \sum_{(j,m) \in i_t^+(m)} q(j,m) \left[\bar{S}(i,t) - \bar{S}(j,m)\right] = 0.$$

Proof. For a given stopping time the problem that must be solved in order to find the lower bound for the value of the American derivative is

$$\max_{\substack{\{\Delta(j,t), B(j,t)\} \\ (j,t) \in \mathcal{J}_t \\ t \in \mathcal{T} \setminus \{T\}}} \Delta_{\tau}(0,0)\, S(0,0) + B_{\tau}(0,0)$$

subject to the following superreplicating conditions

$$\Delta_{\tau}(0,0)\, S(0,0) + B_{\tau}(0,0) \leq G_{\tau}(0,0),$$

if $X_{\tau}(0,0) = 1$, and, otherwise,

$$\Delta_{\tau}(j,t_1)\, S(i,t_2) + B(j,t_1)\, R^{t_2 - t_1} \leq G_{\tau}(i,t_2),$$

for all $t_1 \in \mathcal{T}_m \setminus \{T\}$, $(i,t_2) \in j^+(t_2)$ with $t_2 = \min(s \in \mathcal{T}_m : s > t_1)$, and nodes (i,t_2) such $X_{\tau}(i,t_2) = 1$.

Additionally, for any node (i,t_2), which is a predecessor of a given node (m,t) that satisfies $X_{\tau}(m,t) = 1$, the self-financing conditions apply, *i.e.*,

$$\Delta_{\tau}(k,t_1)\, S(i,t_2) + B_{\tau}(k,t_1)\, R^{t_2 - t_1} \leq \Delta_{\tau}(i,t_2)\, S(i,t_2) + B(i,t_2),$$

where $(i,t_2) \in k_{t_1}^+(t_2)$ and $t_2 = \min(s \in \mathcal{T}_m \setminus \{T\} : s > t_1)$.

Using an analogous procedure as in the proof where the upper bound for the value of the American derivative was found we can write the dual problem of the linear optimization problem described above

$$\min_{q \in Q^{\tau}} \sum\nolimits_{\substack{(i,t) \in \mathcal{J}_t \\ t \in \mathcal{T}_m}} q(i,t)\, \bar{G}_{\tau}(i,t)$$

such that for any (i,t) with $t \in \mathcal{T}_m$

$$\sum_{m>t, m\in\mathcal{T}_m} \sum_{(j,m)\in i_t^+(m)} q(j,m)\left[\bar{S}(i,t) - \bar{S}(j,m)\right] = 0.$$

Optimizing with relation to τ the problem becomes

$$\max_{\tau\in\mathbb{T}} \min_{q\in Q^\tau} \sum\nolimits_{\substack{(i,t)\in\mathcal{J}_t \\ t\in\mathcal{T}_m}} q(i,t)\,\bar{G}_\tau(i,t)$$

such that for any (i,t) with $t \in \mathcal{T}_m$

$$\sum_{m>t, m\in\mathcal{T}_m} \sum_{(j,m)\in i_t^+(m)} q(j,m)\left[\bar{S}(i,t) - \bar{S}(j,m)\right] = 0.$$

■

As for the upper bound the lower bound solving the problem above can also be seen as the solution of a more intuitive problem. In fact, it can be shown that this lower bound maximizes over all possible stopping times the expected discounted payoff, when the expectation is minimized among all adjusted probability measures. In other words,

Theorem 3.4 *There exists an adjusted probability measure $P \in \mathcal{P}(\tau)$ and a stopping time $\tau \in \mathbb{T}$ such that the upper hedging price of an American derivative in a dry market can be written as*

$$V_d^l = \max_{\tau\in\mathbb{T}} \min_{P\in\mathcal{P}(\tau)} E_\tau^p \bar{G}_\tau$$

with $G_\tau(i,t) = X_\tau(i,t)\,G(i,t)$. Additionally, if there is a probability measure with positive probability on every path then the upper hedging price of an American derivative in a dry market can be rewritten as

$$V_d^l = \max_{\tau\in\mathbb{T}} \min_{P\in\mathcal{P}} E^p G_\tau$$

where $G_\tau(i,t) = X_\tau(i,t)\,G(i,t)$, as before.

Proof. Using the result presented in theorem (3.3) and the theorem (2.2) the proof is straightforward. ■

This resulted has already been conjectured as an extension in Harrison and Kreps (1979). When the market is complete then $\mathcal{P}$ is a singleton and the two bounds coincide with the unique arbitrage free value of the American derivative.

In the following section the upper and lower arbitrage free bounds of the American derivatives for the probabilistic case are derived. In the next section, at certain dates, there is uncertainty about the existence of the market.

4 Probabilistic Dry Markets

4.1 The model

As in the previous section we shall work in discrete time, corresponding to dates in $\mathcal{T} = \{0, 1, \ldots, T\}$. Let $\mathcal{T}_m \subseteq \mathcal{T}$ be a set of points in time such that, for all times $t \in \mathcal{T}_m$, transactions are possible with probability one. By assumption, both 0 and T belong to $\mathcal{T}_m$, *i.e.*, transactions are certainly possible at times $t = 0$ and $t = T$. Similarly, let $\mathcal{T}_p \subseteq \mathcal{T}$ be defined as the set of points in time such that transactions are possible, but not certain. For each time $t \in \mathcal{T}_p$, we assume that transactions are possible with an exogenous probability $p > 0$ with $\mathcal{T}_m \cup \mathcal{T}_p = \mathcal{T}$ and $\mathcal{T}_m \cap \mathcal{T}_p = \emptyset$.

We can think of the existence (or not) of the market at time t as the realization of a random variable y_t. This random variable is defined for all $t \in \mathcal{T}$ and it is assumed to be independent of the ordinary source of uncertainty that generates the price process. We can therefore talk about a market existence process. In order to construct one such process, let us first start with the state space. Let $\#(\mathcal{T}_p)$ denote the number of points in $\mathcal{T}_p$. At each of these points, market may either exist or not exist, leading to $2^{\#(\mathcal{T}_p)}$ possible states of nature. We then have the collection of possible states of nature denoted by $\Omega^p = \{v_i\}_{i=1,\ldots,2^{\#(\mathcal{T}_p)}}$, each v_i corresponding to a distinct state. Recall that Ω denotes the set of paths (w) in a perfectly liquid market and $\mathcal{F}_t$ is the σ-algebra generated by the random variable S_t. We now consider the new extended measurable space $\left(\bar{\Omega}, \bar{\mathcal{F}}\right)$, where

$$\bar{\Omega} = \Omega \times \Omega^p$$

and

$$\bar{\mathcal{F}} = \mathcal{F} \times \mathcal{F}^p,$$

with $\mathcal{F}^p = \mathcal{F}_0^p,\ \mathcal{F}_1^p, \ldots,\ \mathcal{F}_T^p$, where $\mathcal{F}_t^p$ is the σ-algebra generated by the random variable y_t. The random variable y_t assumes the values 0 (when there is no market) and 1 (when there is market) and is not dependent on w. Note also that the variable y_t depends only on the information in $\mathcal{F}_p$. Let p_y be the probability associated with the random variable y_t. For all $t \in \mathcal{T}_p$, we have $p_y(y_t = 1) = p$ and $p_y(y_t = 0) = 1 - p$. Similarly, for all $t \in \mathcal{T}_m$, $p_y(y_t = 1) = 1$ and $p_y(y_t = 0) = 0$. Let the $T + 1$ dimensional vector $\mathbf{y}$ denote a given realization of the process $\{y_t\}_{t \in \mathcal{T}}$. There are $2^{\#(\mathcal{T}_p)}$ different possible vectors $\mathbf{y}$.

As in section 2.1, the process followed by the stock price is denoted by S. However, in the presence of probabilistic dry markets the stock price is only observed when market exists, *i.e.*, in all nodes (i, t) such that $y(i, t) = 1$.

As a motivation to what follows, let us consider an example. Consider $\mathcal{T}_m = \{0, 2, 4\}$ and $\mathcal{T}_p = \{1, 3\}$. At $t = 1$ there is a $(1 - p)$ chance that the stock price will not be observed. The same thing happens at $t = 3$. Hence, if there is no new information at these points in time, the σ-algebra describing the information available to the market will be $\mathcal{F}_t = \mathcal{F}_{t-1}$. In our example, there are four different vectors $\mathbf{y}$, given by $\mathbf{y}_1 = (1, 1, 1, 1, 1)$, $\mathbf{y}_2 = (1, 0, 1, 1, 1)$, $\mathbf{y}_3 = (1, 1, 1, 0, 1)$ and $\mathbf{y}_4 = (1, 0, 1, 0, 1)$. Each one is associated with a given probability, respectively, p^2, $p(1-p)$, $p(1-p)$ and $(1-p)^2$. We may describe the trees of information process associated to each of the four possible circumstances as follows

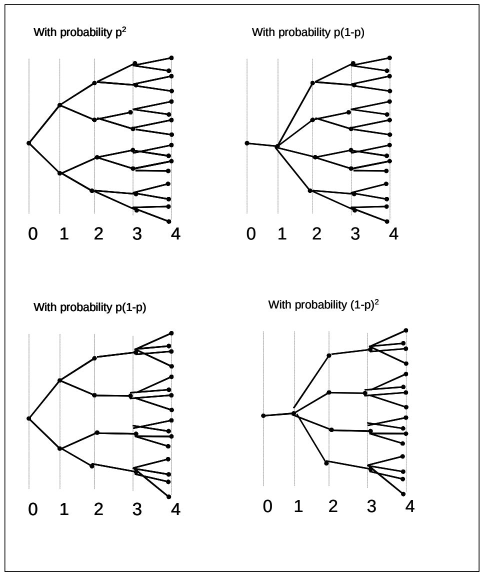

Figure 1: For each $\mathbf{y}$ the information available to the market can be represented by a different tree.

The first tree (top, left) describes the case corresponding to vector $\mathbf{y}_1$, where market exists at all points in time, coinciding with the perfectly liquid market tree. The second tree (top, right) reflects the second case, corresponding to vector $\mathbf{y}_2$, where market does not exist only at $t = 1$. We could have drawn a tree with four branches going directly from the node at $t = 0$ to the corresponding four nodes at $t = 2$. We prefer the representation above, since we want to make clear that the filtration $\mathcal{F}_1$ reflecting the information available at $t = 1$ is the same as the filtration $\mathcal{F}_0$ reflecting the information available at $t = 0$. In a similar way we have a tree representing the vector $\mathbf{y}_3$ (low, left) and another one for the vector $\mathbf{y}_4$ (low, right). However, if we want to describe all the possible situations in the same tree it will look like the one described below

This super-tree plays a main role in the construction of our superhedg-

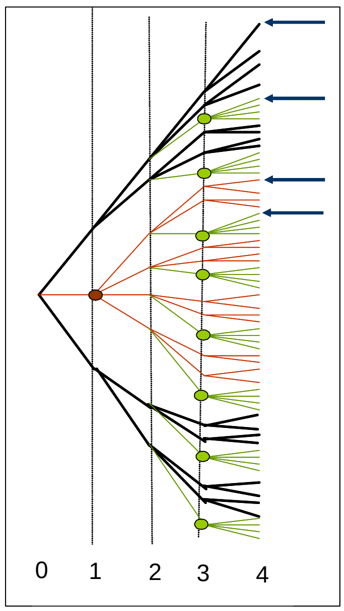

Figure 2: This tree describes all the possibilities under probabilistic illiquidity at $t = 1$ and $t = 3$. The circles identify the nodes when it is not possible to transact. In the final nodes identified with arrows the stock price is the same.

ing strategies. Actually, our extended filtration will work as if we have an extended tree, coinciding with the one above, where transactions would not be permitted at those nodes represented by open circles. We stress the point that nodes in this tree do not represent mere price realizations. They are rather joint representations of the price process and the market existence process. For instance, the terminal nodes indicated with arrows in the figure are assumed to represent the same price level for the underlying asset, but with different market existence realizations.

We now focus on the construction of the superreplicating strategies for probabilistic dry markets. At any point in time, the number of shares and the amount invested in the risk-free asset will depend on the existence, or inexistence, of the market at the previous moments in time. However, these values will not depend on the future existence of the market.

Let $\Delta(j,t;\mathbf{y})$ and $B(j,t;\mathbf{y})$ denote, respectively, the number of shares and the amount invested in the risk free asset at node (j,t) for a given realization, $\mathbf{y}$, of the process $\{y_s\}_{s\in\mathcal{T}}$. We assume that, if $y_t = 0$ and (j,t) is an arbitrary successor of $(i,t-1)$, then $\Delta(j,t;\mathbf{y}) = \Delta(i,t-1;\mathbf{y})$ and $B(j,t;\mathbf{y}) = B(i,t-1;\mathbf{y})$, since the portfolio can not be rebalanced at time t. Hence, for any given two different sets $\mathbf{y}^1$ and $\mathbf{y}^2$ with common values $y_1^1 = y_1^2, y_2^1 = y_2^2, y_3^1 = y_3^2, \ldots$ up to time t, we assume that

$$\Delta\left(j,t;\mathbf{y}^1\right) = \Delta\left(j,t;\mathbf{y}^2\right) \text{ and } B\left(j,t;\mathbf{y}^1\right) = B\left(j,t;\mathbf{y}^2\right).$$

Just as in the deterministic case, let $V(j,t;\mathbf{y})$ denote the value process generated by such portfolio $[\Delta(j,t;\mathbf{y}), B(j,t;\mathbf{y})]$, *i.e.*,

$$V(j,t;\mathbf{y}) = \Delta(j,t;\mathbf{y})\, S(j,t) + B(j,t;\mathbf{y}).$$

Hence,

$$V\left(j,s;\mathbf{y}^1\right) = V\left(j,s;\mathbf{y}^2\right).$$

In an analogous way to the deterministic case, the definition of *self-financed* *strategy* and *superreplicating strategy* is dependent on whether one is in a short or in a long position in the derivative.

In what follows we are going to characterize the upper and lower arbitrage-free bounds for the value of an American derivative in the probabilistic case.

4.2 Some Probabilistic Definitions

Analogously to what we did in section 2.1, we present some mathematical tools to obtain the arbitrage-free bounds of the American derivative.

We begin by defining $\mathcal{T}_{\mathbf{y}}$ as the subset of points in $\mathcal{T}$ after the last non-trading date. Formally we define $\mathcal{T}_{\mathbf{y}} = \{s \in \mathcal{T} : s \geq \Theta(\mathbf{y})\}$ with

$$\Theta(\mathbf{y}) = \begin{cases} 0 \text{ if } y_t = 1, \forall t \in \mathcal{T}, \\ \max(m+1 : y_m = 0) \ \text{ otherwise.} \end{cases}$$

Notice that for liquid markets $\mathcal{T}_{\mathbf{y}} = \mathcal{T}$.

Definition 4.1 *A **node probability measure** is a nonnegative function $q(i,t;\mathbf{y})$ satisfying*

$$\sum\nolimits_{(j,t)} \sum\nolimits_{t \in \mathcal{T}_y} \sum\nolimits_{\mathbf{y}} q(j,t;\mathbf{y}) = 1. \tag{16}$$

Let $Q(\mathbf{y})$ denote the set of all node probability measures $q(i,t;\mathbf{y})$.

Definition 4.2 *A node probability measure $q \in Q(\mathbf{y})$ is said to be a **node martingale measure** if, for any $\mathbf{y}$ and (i,t) such that $t \in \mathcal{T}_y$, satisfies*

$$\begin{aligned} & \sum\nolimits_{\{\mathbf{z}: z_0 = y_0, \ldots, z_t = y_t\}} \sum\nolimits_{\substack{\tau > t \\ \tau \in \mathcal{T}_z}} \sum\nolimits_{(j,\tau) \in \mathbf{i}_t^+(\tau)} q(j,\tau;\mathbf{z}) \bar{S}(i,t) \\ = \ & \sum\nolimits_{\{\mathbf{z}: z_0 = y_0, \ldots, z_t = y_t\}} \sum\nolimits_{\substack{\tau > t \\ \tau \in \mathcal{T}_z}} \sum\nolimits_{(j,\tau) \in \mathbf{i}_t^+(\tau)} q(j,\tau;\mathbf{z}) \bar{S}(j,\tau), \end{aligned}$$

The set of all node martingale measures is denoted by $Q_y^{\mathcal{M}}$.

Definition 4.3 *A node probability measure on the event tree is said to be $\mathbf{y}$-**simple** if, for each $\mathbf{y}$, any t and $t+k \in \mathcal{T}_y$, there are no two nodes in the same path, say (i,t) and $(j,t+k) \in i_t^+(t+k)$, such that $q(i,t;\mathbf{y}) > 0$ and $q(j,t+k;\dot{\mathbf{y}}) > 0$ where $\dot{\mathbf{y}}$ is any set such that $y_1 = \dot{y}_1, \ldots, y_t = \dot{y}_t$.*

The following theorem is analogous to theorem 2.1 but now in the framework of probabilistic dry markets.

Theorem 4.1 (Chalasani and Jha) *The extreme points of the set $Q_y^{\mathcal{M}}$ are simple node probability measures, i.e., on every path on the event tree there is at most one node where q is strictly positive.*

The proof of this theorem follows closely the proof of theorem 2.1.

Definition 4.4 *An **adjusted probability measure** is a nonnegative function $p(i,t;\mathbf{y})$ defined for any $\mathbf{y}$ and (i,t) with $t \in \mathcal{T}_y$ such that $p(0,0) = p(0,0;\mathbf{y}) = 1$ and*

$$p(i,t;\mathbf{y}) = \sum\nolimits_{(j,s)\in i_t^+(s)} \sum\nolimits_{\{\mathbf{z}: z_0=y_0,\ldots,z_t=y_t\}} p(j,s;\mathbf{z}),$$

with $\mathbf{s} = \min\{n \in \mathcal{T}_{\mathbf{z}} : y_n = 1 \text{ and } n > t\}$.

Let the set of all probability measures be denoted by $\mathbb{P}_y$. Also, let τ_y denote an ordinary stopping time that is conditional on the realization of the process $\{y_t\}_{t\in\mathcal{T}}$. For any $\mathbf{y}$, τ_y is a map that is defined from Ω to $\{s \in \mathcal{T} : y_s = 1\}$ such that $\{w : \tau(w;\mathbf{y}) \leq t\} \in \mathcal{F}_t$ for all $t \in \{s \in \mathcal{T} : y_s = 1\}$. Moreover, consider that for two different sets $\mathbf{y}^1$ and $\mathbf{y}^2$ with common values $y_1^1 = y_1^2, y_2^1 = y_2^2, \ldots,$ up to time t, if $\tau\left(w;\mathbf{y}^1\right) = s$, where $s = 0,\ldots,t$, then $\tau\left(w;\mathbf{y}^2\right) = s$. A set of stopping times, one for each $\mathbf{y}$, satisfying the abovementioned property is denoted by τ_Y, i.e.,

$$\tau_Y = \{\tau_y\}_{y\epsilon\left\{y^1,\ldots,\, y^{2^{\#(\mathcal{T}_P)}}\right\}}.$$

Consider $(i,k) \in w$. We define a nonnegative adapted process $X_{\tau,y}$ associated with the stopping time that has the form $X_\tau[i,k;\mathbf{y}] = 1$ if $\tau(w;\mathbf{y}) = k$ and $X_\tau[i,k;\mathbf{y}] = 0$ otherwise. Let $\mathbb{T}_y$ and $\mathbb{X}_{\mathbb{T},y}$ denote the set of all τ_y and associated $X_\tau(\mathbf{y})$, respectively.

Definition 4.5 *A* $\mathbf{y}$*-simple node probability measure is said to be associated with a given stopping time if* $q(i,t;\mathbf{y})$ *is equal to zero when* $X_\tau(i,t;\mathbf{y})$ *is equal to zero, and* $q(i,t;\mathbf{y})$ *is positive when* $X_\tau(i,k;\mathbf{y})$ *is strictly positive, for any* $\mathbf{y}$ *and node* (i,t).

Let the set of all node probability measures with this property be denoted by $Q^\tau(\mathbf{y})$.

Definition 4.6 *For any probability measure* $P_y \in \mathbb{P}_y$ *and stopping time* $\tau_y \in \mathbb{T}_y$ *we say that* P_y *is a* τ_y***-martingale measure*** *if,* P_y*-almost surely, for any* (i,t) *and* $\mathbf{y}$ *such that* $y_t = 1$ *we have*

$$\sum_{(j,s)\in i^+(\mathbf{s})} \sum_{s>t, s\in\mathcal{T}_z} \sum_{\{\mathbf{z}:z_0=y_0,\ldots,z_t=y_t\}} p(j,s;\mathbf{z}) X_\tau(j,s;\mathbf{z}) \left[\bar{S}(i,t) - \bar{S}(j,s)\right] = 0$$

The set of all P_y that have this property is denoted by $\mathcal{P}_y(\tau_y)$

Let $(P_y, X_{\tau,y})$ denote a measure-strategy pair, *i.e.*, a pair constituted by an adjusted probability measure and a nonnegative adapted process.

Definition 4.7 *A measure-strategy pair* $(P_y, X_{\tau,y})$ *is said to be equivalent to a node probability measure if, for any given node* (i,t) *with* $t \in \mathcal{T}_y$, $p(i,t;\mathbf{y}) X_\tau(i,t;\mathbf{y}) = q(i,t;\mathbf{y})$.

We can now enunciate the following result, adapted from Chalasani and Jha (2001) to include the random variable y.

Theorem 4.2 *Consider a node probability measure* $q \in Q(\mathbf{y})$. *Then there exists a measure-strategy pair* (P_y, τ_y) *equivalent to* q*, where for any given* $\mathbf{y}$, P_y *and* $\tau_\mathbf{y}$ *are uniquely defined at node* (i,t) *where*

$$q(i,t;\mathbf{y}) + \sum_{(j,s)\in i^+(\mathbf{s})} \sum_{s>t, s\in\mathcal{T}_z} \sum_{\{\mathbf{z}:z_0=y_0,\ldots,z_t=y_t\}} q(j,s;\mathbf{z})$$

is strictly positive. Conversely, if (P_y, τ_y) *is a measure strategy-pair, then the node function* $q \in Q(\mathbf{y})$ *such that* $q(i,t;\mathbf{y}) = p(i,t;\mathbf{y}) X(i,t;\mathbf{y})$ *is the unique equivalent node-measure.*

Proof. The proof of this theorem is a modification of the one provided in Chalasani and Jha (2001) and analogous to the one of theorem (2.2). ■

All the mathematical definitions provided above are dependent of y. In what follows we will shown that it is possible to define an adjusted probability and a randomized stopping time in the *original tree* that is closely related with the concepts just presented.

Definition 4.8 *An **adjusted probability measure** $\bar{\mathbf{P}}(i,t)$ is a nonnegative function such that $\bar{P}(0,0) = 1$ and $\bar{P}(i,t) = \sum_{(j,t+1)\in i_t^+} \bar{P}(j,t+1)$, for all $t \in \mathcal{T}$.*

The set of all probability measures $\bar{P}$ is denoted by $\bar{\mathbb{P}}$.

A randomized stopping time is a nonnegative adapted process X with the property that on every path of the event tree the sum of the random variable is equal to one, *i.e.*,

$$\sum_{t\in\mathcal{T}} X(i_t, t) = 1 \tag{17}$$

where $i_{t+1} \in i_t^+$. The set of all randomized stopping time is denoted by $\mathbb{X}$.

Definition 4.9 *For a given randomized stopping time $X \in \mathbb{X}$, an adjusted probability measure $\bar{P} \in \bar{\mathbb{P}}$ is said to be a X_y**-martingale measure** if there are a stopping time $X_\tau(y) \in \mathbb{X}_{\mathbb{T},y}$ and a τ_y-martingale measure $P_y \in \mathcal{P}_y(\tau_y)$ such that*

$$X(i,t)\,\bar{P}(i,t) = \sum_{\{\mathbf{y}:y_t=1\}} p(i,t;\mathbf{y})\, X_\tau(i,t;\mathbf{y})$$

for any (i,t) with $t \in \mathcal{T}$.

Let $\bar{\mathcal{P}}(X_y)$ denote the set of all $\bar{P}$ that are X_y martingale measures.

Theorem 4.3 *For any given stopping time $X_\tau(y) \in \mathbb{X}_{\mathbb{T},y}$ and τ_y-martingale measure $P_y \in \mathcal{P}_y(\tau_y)$, there is a randomized stopping time $X \in \mathbb{X}$ and a*

X_y-martingale measure $\bar{P}$, which are defined as follows. Just for notation, let us consider

$$\begin{aligned}\alpha(i,t) &= \sum\nolimits_{r\geq t}\sum\nolimits_{\{\mathbf{z}:z_t=1\}} p(i,r;\mathbf{z})\, X_\tau(i,r;\mathbf{z}) + \\ &\quad \sum\nolimits_{r\geq s}\sum\nolimits_{(j,s)\in i_{t-1}^{+}}\sum\nolimits_{\{\mathbf{z}:z_t\neq 1\}} p(j,r;\mathbf{z})\, X_\tau(j,r;\mathbf{z}),\end{aligned}$$

with $s=\min\{r\in\mathcal{T}: r>t \text{ and } X_\tau(j,r;\mathbf{z})=1\}$.

The adjusted probability measure is such that $\bar{P}(0,0)=1$ and, for any $(i,t)\in j_{t-1}^{+}(t)$ such that $\alpha(j,t-1)\neq 0$,

$$\bar{P}(i,t)=\bar{P}(j,t-1)\frac{\alpha(i,t)}{\sum_{(i,t)\in j_{t-1}^{+}(t)}\alpha(i,t)}.$$

If $\sum_{(i,t)\in j_{t-1}^{+}(t)}\alpha(i,t)=0$ then $\bar{P}(i,t)=\bar{P}(j,t-1)$ for a given successor (i,t) of $(j,t-1)$ and zero for all others successors of $(j,t-1)$.

The randomized stopping time $X\in\mathbb{X}$ is uniquely defined for any node (i,t) such that $\bar{P}(i,t)\neq 0$ and is given by

$$X(i,t)=\frac{\sum_{\{\mathbf{y}:y_t=1\}} p(i,t;\mathbf{y})\, X_\tau(i,t;\mathbf{y})}{\bar{P}(i,t)}.$$

If $\bar{P}(i,t)=0$ but there is a predecessor $(k,t-1)$ such that $\bar{P}(k,t-1)\neq 0$ take

$$X(i,t)=\frac{\sum_{(i,t)\in k_{t-1}^{+}(t)}\alpha(i,t)}{\bar{P}(k,t-1)}.$$

Otherwise, $X(i,t)=0$.

Proof. See appendix B. ■

5 Results on Probabilistic Dry Markets

5.1 Upper bound for the Value of an American Derivative

The upper bound for the value of an American derivative is the maximum value for which the derivative would be transacted without allowing for arbitrage opportunities. As described in the deterministic case, in order to find the upper bound consider a short position in the derivative. The maximum value for which the derivative would be transacted without allowing for arbitrage opportunities would be the value of the cheapest self-financed portfolio that the buyer of the derivative can buy in order to completely hedge against any possibility of exercise of the American derivative.

A strategy is said to be a *self-financed strategy* if for any given $\mathbf{y}$ the portfolio at node (j, t_1), where $t_{\mathbf{1}} \in \{t \in \mathcal{T} : y_t = 1\}$, generates in t_2 a value $\Delta(j, t_1; \mathbf{y}) S(i, t_2) + B(j, t_1; \mathbf{y}) R^{t_2 - t_1}$ such that

$$\Delta(j, t_1; \mathbf{y}) S(i, t_2) + B(j, t_1; \mathbf{y}) R^{t_2 - t_1} \geq V(i, t_2; \mathbf{y}), \tag{18}$$

with $(i, t_2) \in j_{t_1}^{+}(t_2)$ and $t_2 = \min\{s \in \mathcal{T} : s > t \text{ and } y_s = 1\}$.

A sequence of portfolios $\{[\Delta(j, t; \mathbf{y}), B(j, t; \mathbf{y})]\}_{t \in \mathcal{T}}$, one for each $\mathbf{y}$, is said to be a *superreplicating strategy* if the value of each portfolio is higher than or equal to the payoff of the derivative at any node in the next transaction time. In other words, for any trading dates t_1 and t_2 such that $t_1 \in \{t \in \mathcal{T} : y_t = 1\}$ and $t_2 = \min\{t \in \mathcal{T} : t > t_1 \text{ and } y_t = 1\}$ and arbitrary nodes, (j, t_1) and $(i, t_2) \in j_{t_1}^{+}(t_2)$, the portfolio at t_1, $[\Delta(j, t_1; \mathbf{y}), B(j, t_1; \mathbf{y})]$, must be such as to generate in t_2 a value $\Delta(j, t_1; \mathbf{y}) S(i, t_2) + B(j, t_1; \mathbf{y}) R^{t_2 - t_1}$ such that

$$\Delta(j, t_1; \mathbf{y}) S(i, t_2) + B(j, t_1; \mathbf{y}) R^{t_2 - t_1} \geq G(i, t_2). \tag{19}$$

Since it is the cheapest initial portfolio, the upper bound V_p^u must satisfy

$$V_p^u = \min V(0, 0).$$

The decision variables are the $\Delta(j, t; \mathbf{y})$ and $B(j, t; \mathbf{y})$ for all non-terminal nodes of the event tree. However, this optimization is subject to the constraints of self-financing (18) and superreplication (19).

More formally, for any given $\mathbf{y}$ take any $t_1 \in \mathcal{T}$ such that $y_{t_1} = 1$. Define the consecutive trading date t_2 as $t_2 = \min(s \in \mathcal{T} : s > t_1 \text{ and } y_s = 1)$. The upper bound for the value of the American derivative can thus be seen as the solution of the following problem:

$$V_p^u = \min_{\{\Delta(j,t;\mathbf{y}),B(j,t;\mathbf{y})\}_{t\in\{s\in\mathcal{T}:y_s=1\}\setminus\{T\}}} \Delta(0,0)\,S(0,0)+B(0,0)$$

subject to the superreplicating constraints:

$$\Delta(0,0)\,S(0,0)+B(0,0)\geq G(0,0)\,, \tag{20}$$

$$\Delta(j,t_1;\mathbf{y})\,S(i,t_2)+B(j,t_1;\mathbf{y})\,R^{t_2-t_1}\geq G(i,t_2)\,, \tag{21}$$

and subject to the self-financing constraints:

$$\Delta(j,t_1;\mathbf{y})\,S(i,t_2)+B(j,t_1;\mathbf{y})\,R^{t_2-t_1}\geq\Delta(i,t_2;\mathbf{y})\,S(i,t_2)+B(i,t_2;\mathbf{y}) \tag{22}$$

for any $(i,t_2)\in j_{t_1}^{+}(t_2)$.

Using results from linear programming the upper bound arbitrage free bound of the American derivative can be written as follows.

Theorem 5.1 *There is a node probability measure $q\in Q(y)$ such that the upper hedging price of an American derivative in a probabilistic dry market can be written as*

$$V_p^u=\max_{q\in Q_y^{\mathcal{M}}}\sum\nolimits_{(j,t)}\sum\nolimits_{t\in\mathcal{T}_y}\sum\nolimits_{\mathbf{y}}q(j,t;\mathbf{y})\,\bar{G}(j,t)\,.$$

Proof. This proof follows the methodology used in theorem (3.2). As the upper bound for the value of the American derivative, V_p^u, is the solution of linear programming problem it is possible to construct its dual. Let $\lambda(0,0)$, $\lambda(i,t_2;\mathbf{y})$ and $\gamma(i,t_2;\mathbf{y})$ denote the dual variables that are associated, respectively, with the constraints (20), (21) and (22) of the primal problem. Note that, as we assume that given two different sets $\mathbf{y}^1$ and $\mathbf{y}^2$ with common values $y_1^1=y_1^2, y_2^1=y_2^2, y_3^1=y_3^2,\ldots$ up to time t_1, the portfolio will be the same, *i.e.*,

$$\Delta\left(j,t_1;\mathbf{y}^1\right)=\Delta\left(j,t_1;\mathbf{y}^2\right)\text{ and }B\left(j,t_1;\mathbf{y}^1\right)=B\left(j,t_1;\mathbf{y}^2\right).$$

Then, $\lambda\left(i,t_2;\mathbf{y}^1\right) = \lambda\left(i,t_2;\mathbf{y}^2\right)$ for all $(i,t_2) \in i_{t_1}^{+}(t_2)$. Before presenting the dual problem let define

$$\mathbf{\Theta}_t = \{\mathbf{y} : y_t = 1 \text{ and } \min\left[s \in \mathcal{T}_y : s > t\right] = \min\left[s \in \mathcal{T} : y_s = 1 \text{ and } s > t\right]\}.$$

The dual problem is given by

$$\max_{q \in Q(y)} \sum\nolimits_{(j,t)} \sum\nolimits_{t \in \mathcal{T}_y} \sum\nolimits_{\mathbf{y}} \lambda\left(j,t;\mathbf{y}\right) G\left(j,t\right)$$

subject to the conditions:

$$\lambda\left(0,0\right) S\left(0,0\right) + \sum\nolimits_{(j,s) \in i_0^{+}(t)} \sum\nolimits_{z \in \mathbf{\Theta}_0} \left[\lambda\left(j,t;\mathbf{z}\right) + \gamma\left(j,t;\mathbf{z}\right)\right] S\left(j,t\right) = S\left(0,0\right), \tag{23}$$

$$\lambda\left(0,0\right) + \sum\nolimits_{(j,s) \in i_0^{+}(t)} \sum\nolimits_{\mathbf{z} \in \mathbf{\Theta}_0} \left[\lambda\left(j,t;\mathbf{z}\right) + \gamma\left(j,t;\mathbf{z}\right)\right] R^t = 1 \tag{24}$$

where $t = \min\left\{s \in \mathcal{T}_z : s > 0\right\}$.

For any (i,t) and $\mathbf{y}$ such that $t \in \mathcal{T}_y \setminus \left\{0, \max\left[r \in \mathcal{T}_y \text{ and } r < T\right], T\right\}$,

$$\sum\nolimits_{(j,s) \in i_t^{+}(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \left[\lambda\left(j,s;\mathbf{z}\right) + \gamma\left(j,s;\mathbf{z}\right)\right] S\left(j,s\right) = \gamma\left(i,t;\mathbf{y}\right) S\left(i,t\right) \tag{25}$$

and

$$\sum\nolimits_{(j,s) \in i_t^{+}(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \left[\lambda\left(j,s;\mathbf{z}\right) + \gamma\left(j,s;\mathbf{z}\right)\right] R^{s-t} = \gamma\left(i,t;\mathbf{y}\right) \tag{26}$$

where $s = \min\left\{r \in \mathcal{T}_z : r > t\right\}$. Finally, for any (i,t) and $\mathbf{y}$ such that $t = \max\left\{r \in \mathcal{T}_y \text{ and } r < T\right\}$

$$\sum\nolimits_{(j,T) \in i_t^{+}(T)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda\left(j,T;\mathbf{z}\right) S\left(j,T\right) - \gamma\left(i,t;\mathbf{y}\right) S\left(i,t\right) = 0 \tag{27}$$

and

$$\sum\nolimits_{(j,T) \in i_t^{+}(T)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda\left(j,T;\mathbf{z}\right) R^{T-t} - \gamma\left(i,t;\mathbf{y}\right) = 0 \tag{28}$$

The constraints presented in equations (25) and (27) can be rewritten as

$$S\left(i,t\right) \gamma\left(i,t\right) = \sum\nolimits_{r \geq s} \sum\nolimits_{(j,r) \in i_t^{+}(r)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda\left(j,r;\mathbf{z}\right) S\left(j,r\right)$$

and the constraints presented in equations (26) and (28) can be rewritten as

$$\gamma(i,t) = \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) R^{r-t}.$$

for all (i,t), $t \in \mathcal{T} \backslash \{0,T\}$ and with $s = \min\{r \in \mathcal{T}_z : r > t\}$. The two previous equations can be written as

$$\begin{aligned} & S(i,t) \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) R^{r-t} \\ = \; & \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) S(j,r) \end{aligned}$$

Taking into account equations (23) and (24) we obtain, for all $t \in \mathcal{T} \backslash \{T\}$

$$\begin{aligned} & S(i,t) \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) R^{r-t} \\ = \; & \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) S(j,r) \end{aligned}$$

with

$$\sum\nolimits_{t \in \mathcal{T}_y} \sum\nolimits_{(j,t)} \sum\nolimits_{\mathbf{y}} \lambda(j,t;\mathbf{y}) R^t = 1.$$

Let $q(i,t;\mathbf{y}) = \lambda(i,t;\mathbf{y}) R^t$. Then,

$$\begin{aligned} & \bar{S}(i,t) \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) R^{r-t} \\ = \; & \sum\nolimits_{r \geq s} \sum\nolimits_{(j,s) \in i_t^+(s)} \sum\nolimits_{z \in \mathbf{\Theta}_t} \lambda(j,r;\mathbf{z}) \bar{S}(j,r) \end{aligned}$$

and

$$\sum\nolimits_{t \in \mathcal{T}_y} \sum\nolimits_{(j,t)} \sum\nolimits_{\mathbf{y}} q(j,t;\mathbf{y}) = 1.$$

■

The upper bound solving the problem above can also be seen as the solution of a more intuitive problem. In fact, it can be shown that this upper bound maximizes over all possible stopping times the expected discounted payoff, when the expectation is optimized among all adjusted probability measures. In other words,

Theorem 5.2 *There is an adjusted probability measure $P_y \in \mathcal{P}_y(\tau_y)$ and an adapted process $X_{\tau,y} \in \mathbb{X}_{\mathbb{T},y}$ such that the upper hedging price of an American derivative in a probabilistic dry market can be written as*

$$V_p^u = \max_{X_{\tau,y} \in \mathbb{X}_{\mathbb{T},y}} \max_{P_y \in \mathcal{P}_y(\tau_y)} E^{\bar{P}} G_{X_{\tau,y}}$$

where $G_{X_{\tau,y}}(i,t) = G(i,t)\, X_{\tau,y}(i,t)$

Proof. In an analogous way to the proof of theorem (3.2), using theorem (4.1) and theorem (4.2) the conclusion is straightforward. ■

Note that this result is the same that would be obtained if the filtration that describes the stock price is an augmented one, in the spirit of the one presented in figure (2), with no uncertainty about the existence of the market and no transactions in some nodes (the ones identified in the figure). And, using this filtration, we found out that ordinary stopping times are enough to write the upper bound as an expectation.

However, the upper bound of the value of an American derivative can also be written using randomized stopping times if an adjusted probability measure with an additional characteristic is considered. The adjusted probability measure have to be decomposed in such a way that if an augmented filtration is considered the stock price is a martingale.

If the initial filtration is considered it is not possible to write the upper bound as an optimization over ordinary stopping times, as in theorem (5.2). In this case, randomized stopping times may be needed.

Theorem 5.3 *There is an adjusted probability measure $\bar{P} \in \bar{\mathcal{P}}(X_y)$ and a process $X \in \mathbb{X}$ such that the upper hedging price of an American derivative in a probabilistic dry market can be written as*

$$V_p^u = \max_{X \in \mathbb{X}} \max_{\bar{P} \in \bar{\mathcal{P}}(X_y)} E^{\bar{P}} G_X$$

with $G_X(i,t) = G(i,t)\, X(i,t)$.

Proof. In order to prove that the optimum value determined by the optimization problems in Theorem 5.3 coincide with the one presented in Theorem 5.2, we begin by noticing that

$$\begin{array}{c} X\left(i,t\right)\bar{P}(i,t)=\sum_{\{\mathbf{y}:y_t=1\}}p\left(i,t;\mathbf{y}\right)X_{\tau}\left(i,t;\mathbf{y}\right) \\ \Rightarrow \\ \sum_{t\in\mathcal{T}}\sum_{(i,t)\in\mathcal{J}_t}X\left(i,t\right)\bar{P}(i,t)G\left(i,t\right)= \\ =\sum_{t\in\mathcal{T}_y}\sum_{(i,t)\in\mathcal{J}_t}\sum_{\mathbf{y}}p\left(i,t;\mathbf{y}\right)X_{\tau}\left(i,t;\mathbf{y}\right)G\left(i,t\right) \\ \Leftrightarrow \\ E^{\bar{P}}G_X=E^{P_y}G_{X_{\tau,y}}. \end{array} \tag{29}$$

Then, we establish a relation between the two sets over which the optimization problems presented in the two theorems are performed. In order to establish this relation, we begin by considering an element P_y of $\cup_{X_{\tau,y}\in\mathbb{X}_{\mathbb{T},y}}P_y\left(\tau_y\right)$. By theorem 4.3 and the implication presented in equations (29), for any element P_y of $\cup_{X_{\tau,y}\in\mathbb{X}_{\mathbb{T},y}}P_y\left(\tau_y\right)$,there exists an element $\bar{P}$ that belongs to $\cup_{X\in\mathbb{X}}\bar{P}\left(X_y\right)$, such that $E^{\bar{P}}G_X=E^{P_y}G_{X_{\tau,y}}$. Now, consider an element of $\cup_{X\in\mathbb{X}}\bar{P}\left(X_y\right)$. By the definition of a X_y-martingale measure and, once again, by the relation presented in equations (29), for any element $\bar{P}$ in $\cup_{X\in\mathbb{X}}\bar{P}\left(X_y\right)$ there exists an element P_y in $\cup_{X_{\tau,y}\in\mathbb{X}_{\mathbb{T},y}}P_y\left(\tau_y\right)$ such that $E^{\bar{P}}G_X=E^{P_y}G_{X_{\tau,y}}$. Hence, given the relation established between the two sets the values determined by the two optimization problems coincide.. ■

In what follows we are going to consider an example. The upper bound of the American derivative is obtained using the primal and the dual problem. In this example no optimal pure stopping time exists that maximizes the expected value of the payoffs of the American derivative. The expected value of the payoffs of the American derivative is maximized with randomized stopping times.

Consider $T=\{0,1,2\}$, $\mathcal{T}=\{0,2\}$ and $\mathcal{T}_p=\{1\}$. Let $R=1$ and the uncertainty about the price of the underlying stock and the derivative be given by

There are two sets $\mathbf{y}$, $\mathbf{y}_1=\{1,1,1\}$ and $\mathbf{y}_2=\{1,0,1\}$. The optimum

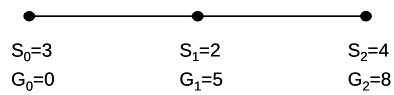

Figure 3: The evolution of the price of the underlying asset and the derivative´s payoff.

value of the variables in the primal problem is

$$\begin{array}{l} \Delta(0,0;\mathbf{y}_1) = \Delta(0,0;\mathbf{y}_2) = \Delta(0,0) = 1,5 \\ B(0,0;\mathbf{y}_1) = B(0,0;\mathbf{y}_2) = B(0,0) = 2 \\ \Delta(0,1;\mathbf{y}_1) = 2,5 \\ B(0,1;\mathbf{y}_1) = 0 \end{array}$$

that results in an optimum value of the function

$$\Delta(0,0)\, S(0,0) + B(0,0) = 6,5.$$

In what concerns the dual problem the optimum value of the variables

$$\begin{array}{l} q(0,0) = q(0,0;\mathbf{y}_1) = q(0,0;\mathbf{y}_2) = 0 \\ q(0,1) = q(0,1;\mathbf{y}_1) = 0.5 \\ q(0,2) = q(0,2;\mathbf{y}_1) + q(0,2;\mathbf{y}_2) = 0 + 0.5 = 0.5 \end{array}$$

As a result, the optimum value of the objective function is

$$q(0,1)\, G(0,1) + q(0,2)\, G(0,2) = 6.5.$$

In this case the probability measure P_y is given by

$$\begin{array}{l} p_y(0,0) = 1 \\ p_y(0,1,\mathbf{y}_1) = 1 * \frac{0.5}{0.5+0.5} = 0.5 \\ p_y(0,2,\mathbf{y}_1) = 0.5 \\ p_y(0,2,\mathbf{y}_2) = 0.5 \end{array}$$

and the stopping time τ_y is such that X_τ is given by

$$\begin{array}{l} X_\tau(0,0) = 0 \\ X_\tau(0,1,\mathbf{y}_1) = 1 \\ X_\tau(0,2,\mathbf{y}_1) = 0 \\ X_\tau(0,2,\mathbf{y}_2) = 1 \end{array}$$

The probability measure $\bar{P}$ is

$$P(0,0) = P(0,1) = P(0,2) = 1$$

and the randomized stopping time X

$$\begin{array}{l} X(0,0) = 0 \\ X(0,1) = 0.5 \\ X(0,2) = 0.5 \end{array}$$

As described in the example, the use of the randomized or ordinary stopping times is closed related to the filtration that is being used.

The point that explains the difference between our result and that of Chalasani and Jha (2001) is the following. Under no transaction costs and complete markets, there is only one node per path such that the value of the superhedging portfolio fully replicates the derivative's payoff. This unique node per path would correspond to the optimal exercise of the derivative.

In the setting of Chalasani and Jha (2001), rebalancing the superhedging portfolio is possible at any point in time, and the derivatives have well defined payoffs at any point in time. However, due to transaction costs, it may be optimal for their problem not to rebalance at *some* points in time. The cheapest superhedging strategy could then be to replicate the derivative's payoff in consecutive points in time, for a given path. These points with full replication correspond to optimal stopping. Since there may be more than one per path, the optimal stopping time would be randomized.

In our case it is not possible to exercise the derivative when there is no market for the underlying asset, and hence there is no need to hedge for exercise at those points where it is not possible to rebalance the portfolio. In particular, in the case of probabilistic dry markets, our representation of the superreplicating bounds with deterministic stopping times is strongly driven by the fact that we consider the enlarged filtration resulting from the price process and the market-existence process. This enlarged filtration allows for at most one node, per path, such that the value of the superhedging portfolio fully replicates the derivative's payoff, avoiding this way the randomized stopping times. If that were not the case, the resulting stopping times could also be randomized. In fact, had we considered only the filtration generated by the price process, for any given price path it could be optimal fully replicate the derivative's payoff at different moments in time.

5.2 Lower bound for the Value of an American Derivative

The lower bound for the value of an American derivative is the minimum value for which the derivative would be transacted without allowing for arbitrage opportunities. As in the deterministic case, in order to find the lower bound consider a long position in the derivative. For a given exercise policy consider the most expensive self-financed portfolio that the buyer of the American derivative can buy in order to be fully hedged. The minimum value for which the derivative would be transacted without allowing for arbitrage opportunities would be the value of most expensive portfolio chosen among all the portfolios just mentioned.

For any given stopping time τ_y consider any node (j,t), such that (j,t) is a predecessor of (k,m), with $X_\tau(k,m;\mathbf{y})=1$. Let the set of (k,m) and all its predecessors be denoted by $J^-_{\tau_y}$, i.e.,

$$\begin{aligned} J^-_{\tau_y} &= \{(j,t) : (j,t) \text{ is a predecessor of } (k,m), \text{with } X_\tau(k,m;\mathbf{y})=1\} \\ &\quad \cup \{(k,m) : X_\tau(k,m;\mathbf{y})=1\}. \end{aligned}$$

For any given stopping time τ_y consider the portfolio constituted of $\Delta^{\tau_y}(j,t;\mathbf{y})$ shares of the underlying asset and an amount $B^{\tau_y}(j,t;\mathbf{y})$ invested in the risk free asset, with $(j,t)\in J^-_{\tau_y}$. Note that if $y_t=0$ and (j,t) is an arbitrary successor of $(i,t-1)$, then $\Delta^{\tau_y}(j,t;\mathbf{y})=\Delta^{\tau_y}(i,t-1;\mathbf{y})$ and $B^{\tau_y}(j,t;\mathbf{y})=B^{\tau_y}(i,t-1;\mathbf{y})$, since the portfolio can not be rebalanced at time t. Moreover, for any given two different sets $\mathbf{y}^1$ and $\mathbf{y}^2$ with common values $y_1^1 = y_1^2, y_2^1 = y_2^2, y_3^1 = y_3^2, \ldots$ up to time t, it was assumed that

$$\Delta^{\tau_y}\left(j,t;\mathbf{y}^1\right) = \Delta^{\tau_y}\left(j,t;\mathbf{y}^2\right) \text{ and } B^{\tau_y}\left(j,t;\mathbf{y}^1\right) = B^{\tau_y}\left(j,t;\mathbf{y}^2\right).$$

The value process of the portfolio $\left[\Delta^{\tau_y}(j,t;\mathbf{y}), B^{\tau_y}(j,t;\mathbf{y})\right]$ is given by

$$V^{\tau_y}(i,t_2;\mathbf{y}) = \Delta^{\tau_y}(i,t_2;\mathbf{y})\, S(i,t_2) + B^{\tau_y}(i,t_2;\mathbf{y}).$$

For each τ_y consider the set of portfolios $\left[\Delta^{\tau_y}(j,t;\mathbf{y}), B^{\tau_y}(j,t;\mathbf{y})\right]_{\forall (j,t)\in J^-_{\tau_y}}$, one for each node that is a predecessor of the node where X_τ is equal to one. Let it be denoted by $\left[\Delta^{\tau_y}, B^{\tau_y}\right]$, *i.e.*,

$$\left[\Delta^{\tau_y}, B^{\tau_y}\right] = \left[\Delta^{\tau_y}(j,t;\mathbf{y}), B^{\tau_y}(j,t;\mathbf{y})\right]_{\forall (j,t)\in J^-_{\tau_y}}.$$

The set of these set of portfolios, one for each $\tau_y \in \tau_Y$ be denoted by $\left[\Delta^{\tau_Y}, B^{\tau_Y}\right]$, *i.e.*,

$$\left[\Delta^{\tau_Y}, B^{\tau_Y}\right] = \left[\Delta^{\tau_y}(j,t;\mathbf{y}), B^{\tau_y}(j,t;\mathbf{y})\right]_{\forall (j,t)\in J^-_{\tau_y}, \tau_y\in\tau_Y}.$$

For a long position in the derivative, a set of portfolios that belongs to $\left[\Delta^{\tau_Y}, B^{\tau_Y}\right]$ is said to be a *self-financed strategy* if for any nodes (j,t_1) and (i,t_2) that are predecessors of the node (m,t_3), such that $X_\tau(m,t_3) = 1$,

$$\Delta^{\tau_y}(j,t_1;\mathbf{y})\, S(i,t_2) + B^{\tau_y}(j,t_1;\mathbf{y})\, R^{t_2-t_1} \le V^{\tau_y}(i,t_2;\mathbf{y}), \tag{30}$$

with $t_{\mathbf{1}} \in \{t \in \mathcal{T} : y_t = 1\}$, $t_2 = \min\{s \in \mathcal{T} : s > t_1 \text{ and } y_s = 1\}$ and $(i,t_2) \in j^+_{t_1}(t_2)$.

A set of portfolios that belongs to $\left[\Delta^{\tau_Y}, B^{\tau_Y}\right]$, is said to be a *superreplicating strategy* if the value of each portfolio is lower than or equal to the payoff of the derivative at any node in the next transaction time. In other words, for any trading dates t_1 and t_2 such that $t_1 \in \{t \in \mathcal{T} : y_t = 1\}$ and $t_2 = \min\{t \in \mathcal{T} : t > t_1 \text{ and } y_t = 1\}$ and arbitrary nodes, (j,t_1) and $(i,t_2) \in j^+_{t_1}(t_2)$ such that $X_\tau(i,t_2) = 1$, the portfolio at t_1, $\left[\Delta^{\tau}(j,t_1;\mathbf{y}), B^{\tau}(j,t_1;\mathbf{y})\right]$, must be such as to generate in t_2 a value such that

$$\Delta^{\tau_y}(j,t_1;\mathbf{y})\, S(i,t_2) + B^{\tau_y}(j,t_1;\mathbf{y})\, R^{t_2-t_1} \le G(i,t_2). \tag{31}$$

For each τ_Y consider the most expensive portfolio that is self-financing and superreplicating, *i.e.*, the most expensive portfolio that respects conditions (30) and (31), respectively. Let this portfolio be denoted by $V_{l,p}^{\tau_Y}$. The lower bound V_p^l must satisfy

$$V_p^l = \max_{\tau_Y} V_{l,p}^{\tau_Y}(0,0).$$

More formally, the lower bound for the value of the American derivative can thus be seen as the solution of the following problem:

$$V_p^l = \max_{\tau_Y} \max_{\{[\Delta^{\tau_y}(j,t;\mathbf{y}),B^{\tau_y}(j,t;\mathbf{y})]\}_{(j,t)\in J_{\tau_y}^-}\epsilon\left[\Delta^{\tau_Y},B^{\tau_Y}\right]} \Delta^{\tau_y}(0,0)\,S(0,0)+B^{\tau_y}(0,0)$$

subject to the superreplicating constraint

$$\Delta^{\tau_y}(0,0)\,S(0,0) + B^{\tau_y}(0,0) \leq G(0,0)\,,$$

if $X_{\tau_y}(0,0) = 1$. However, if $X_{\tau_y}(0,0) = 0$, the superreplication condition is defined for any node (i,t_2) such that $X(i,t_2) = 1$, and is given by

$$\Delta^{\tau_y}(j,t_1)\,S(i,t_2) + B^{\tau_y}(j,t_1)\,R^{t_2-t_1} \leq G(i,t_2)\,,$$

for any $t_1 \in \mathcal{T}_m \backslash \{T\}$ such that $(i,t_2) \in j^+(t_2)$ and $t_2 = \min(s \in \mathcal{T}_m : s > t_1)$.

Additionally, for any node (m,t_3) such that $X_{\tau_y}(m,t_3) = 1$ the self-financing conditions apply, *i.e.*,

$$\Delta^{\tau_y}(j,t_1;\mathbf{y})\,S(i,t_2) + B^{\tau_y}(j,t_1;\mathbf{y})\,R^{t_2-t_1} \leq V^{\tau_y}(i,t_2;\mathbf{y})\,,$$

for any (i,t_2) such that $(m,t_3) \in i_{t_2}^+(t_3)$, for any (j,t_1) such that $(i,t_2) \in j_{t_1}^+(t_2)$ with $t_{\mathbf{1}} \in \{t \in \mathcal{T} : y_t = 1\}$ and $t_2 = \min\{s \in \mathcal{T} : s > t_1 \text{ and } y_s = 1\}$.

Let us consider the worse scenario in what concerns the existence of the market, that is, the situation that corresponds to the $\mathbf{y}$ with the highest number of zeros. This situation corresponds to deterministic dryness. Let this $\mathbf{y}$ be denoted by $\mathbf{y}_d$.

Theorem 5.4 *The lower hedging price of an American derivative in a probabilistic dry market coincides with the one of the deterministic case.*

Proof. The restrictions of the maximization problem that must be solved to find the lower bound in the deterministic case are a subset of the ones presented in the probabilistic case. The restriction of the deterministic case corresponds to $\mathbf{y} = \mathbf{y}_d$, in the probabilistic case, for any possible τ_Y. Therefore, $V_d^l \geq V_p^l$. Moreover, as the solution of the deterministic case respects the restrictions of the probabilistic case $V_d^l = V_p^l$. It corresponds to choose, for any $(j,t) \in J_{\tau_y}^-$, $\tau_{\mathbf{y}_i} = \tau_{\mathbf{y}_d}$, $\Delta^{\tau_y}(j,t;\mathbf{y}) = \Delta^{\tau_{y_d}}(0,0;\mathbf{y}_d)$ and $B^{\tau_y}(j,t;\mathbf{y}) = B^{\tau_{y_d}}(0,0;\mathbf{y}_d)$. ■

6 Comparison of the Results

In this section we will compare the arbitrage-free bounds of an American derivative in a deterministic dry market, in a probabilistic dry market and in a market where transactions are possible at any point in time. In other words, we will compare the arbitrage-free bounds of an American derivative if, at some given points in time, transactions are not possible, transactions are possible with a given probability and transactions are certain.

The upper bound in a probabilistic dry market is higher than or equal to the upper bound if the market is dry in the deterministic sense. Moreover, it is also equal to or higher than the upper bound if transactions were possible at all points in time (V^u). The reason is that we are using the *pure* arbitrage-free concept. If, at a given point in time, it becomes possible to transact with a given probability, the seller of the American derivative must hedge against the possibility of exercise at that point in time. The value of the probability is irrelevant because he will hedge against the worse scenario. In what concerns the upper bound in a deterministic dry market it can be smaller or higher than the upper bound if transactions were possible at all points in time. The reason for this is quite intuitive. Consider an American derivative with a very high payoff in a given moment where transactions were not possible due to the deterministic dryness. If transactions were possible at that given moment in time, the value of the American derivative could increase to become higher than the upper bound in a deterministic dry market. Summing up, $V_p^u \geq V_d^u$, $V_p^u \geq V^u$ and $V_d^u \lesseqgtr V^u$.

The lower bound in a probabilistic dry market is equal to the lower bound if the market is dry in the deterministic sense. Moreover, it is also lower than or equal to the lower bound if transactions were possible at all points in time (V^l). The reason is as follows. The lower bound is the value of the most expensive portfolio that is self-financed and superreplicates the payoffs of the derivative that is being bought, *i.e.*, at the exercise date its value is smaller than, or equal to, the payoff of the derivative that we are receiving. If a given point is not possible to transact and then becomes possible to transact with a given probability several constraints, which concern the exercise at this additional date, are added to the problem that characterizes the lower bound. Hence, $V_p^l \leq V_d^l$. However, as the solution of the deterministic case is a possible solution of the probabilistic case we conclude that $V_p^l = V_d^l$. In what concerns the comparison with the case where transactions are possible at all points in time we have $V_p^l \lesseqgtr V^l$. The reason is that the constraints of the problem that characterizes the lower bound when transactions are possible at all points in time are a subset of the ones presented in the probabilistic case.

If the market is incomplete even with the existence of transactions at all points in time it is not possible to find a unique arbitrage free value for the American derivative. However, it is also possible to establish an arbitrage free range of variation for the value of the American derivative. This range will be a subset of the arbitrage free range of variation for the value of the American derivative in the case of probabilistic dryness, but may not be a subset of the arbitrage free range of variation in the deterministic case.

Considering that the only source of incompleteness in the market is the non-existence, or the possibility of non-existence, of the market at some points in time. If transactions were possible at all points in time, markets would be complete and there would be a unique arbitrage-free value for any American derivative. We found out that this unique arbitrage-free value for each American derivative belongs to the arbitrage-free range of variation for its value under a probabilistic dry market. However, it may not belong to the arbitrage-free range if a deterministic dry market is considered.

7 Exercise Policy

In order to understand the optimal exercise policy, we start presenting the case of a complete market, followed by the case of incomplete markets.

7.1 Complete Markets

In the case of complete markets, the value of an American derivative is given by

$$V^u = \max_{\tau \in \mathbb{T}} \max_{P \in \mathcal{P}} E^p G_\tau \tag{32}$$

where $G_\tau (i,t) = X_\tau (i,t) \, G(i,t)$, as before.

If the solution is unique, the stopping time that solves (32) is the optimal exercise policy for the holder of the American derivative.

Let us analyze this result in some detail. Given an optimal stopping time τ^*, we may define a *stopping time frontier* as follows.

Definition 7.1 *A stopping time frontier is the set of nodes* (i,t) *such that* $X_{\tau^*} (i,t) = 1$.

Recalling that there is an optimal stopping node for each possible path[2],

[2] If the solution is unique, there is a unique strictly positive q associated to each path. Hence, the stopping time is uniquely defined.

we define the *interior of the stopping time frontier* as follows.

Definition 7.2 *The interior of the stopping time frontier is the set of predecessors of the stopping time frontier.*

It follows that no rational agent exercises the American derivative at a node inside the stopping time frontier, because at such nodes, the American derivative is worth more than the corresponding exercise. A rational agent would exercise the American derivative whenever the stopping time frontier is reached. This happens because the derivative's payoff at that point is larger than the cost of a replicating portfolio, guaranteeing the derivative's payoff in the future.

If the solution is not unique, there may be indeterminacy, even in this case of complete markets. An example illustrates this point. Consider the non-terminal node (i, t_1) and two immediate successors of (i, t_1), nodes (j, t_2) and (m, t_2). The replicating portfolio, at node (i, t_1), is the pair $[\Delta(i, t_1), B(i, t_1)]$. Assume that this portfolio satisfies

$$\begin{array}{c} \Delta(i, t_1) S(j, t_2) + B(i, t_1) R = G(j, t_2) \\ \Delta(i, t_1) S(m, t_2) + B(i, t_1) R = G(m, t_2) \end{array} \tag{33}$$

We also assume that, at node (i, t_1),

$$G(i, t_1) = V(i, t_1). \tag{34}$$

Moreover, let $[\Delta(j, t_2), B(j, t_2)]$ and $[\Delta(m, t_2), B(m, t_2)]$ denote the superreplicating portfolios at nodes (j, t_2) and (m, t_2), respectively. We also assume that

$$G(j, t_2) > V(j, t_2) \text{ and } G(m, t_2) > V(m, t_2)$$

In this case, the value of the portfolio, at node (i, t_1), that replicates the value of the American derivative in nodes (j, t_2) and (m, t_2) is the same as the payoff of the American derivative. Let $P(i, t_1)$ denote the price of the American derivative at node (i, t_1). In this case $P(i, t_1) = G(i, t_1) = V(i, t_1)$. Hence, at node (i, t_1) the holder of the American derivative will obtain the same payoff by either exercising or selling the derivative. However, when node (j, t_2), or (m, t_2), is reached the American derivative will be exercised.

Since the replicating portfolio satisfies (33) and (34), the solution of the dual problem is not unique. There are several node probability measures q solving the maximization problem that characterizes the value of the derivative. Let q_1 and q_2 denote two possible solutions. In that case, q_1 and q_2 must satisfy

$$V = \max_{q_1 \in Q} \sum\nolimits_{\substack{(i,t) \in \mathcal{J}_t \\ t \in \mathcal{T}_m}} q_1(i,t)\, \bar{G}(i,t) = \max_{q_2 \in Q} \sum\nolimits_{\substack{(i,t) \in \mathcal{J}_t \\ t \in \mathcal{T}_m}} q_2(i,t)\, \bar{G}(i,t)$$

such that for any (i, t) with $t \in \mathcal{T}_m$

$$\begin{aligned} \sum_{m>t, m \in \mathcal{T}_m} \sum_{(j,m) \in i_t^+(m)} q_1(j,m) \left[\bar{S}(i,t) - \bar{S}(j,m)\right] &= 0 \\ \sum_{m>t, m \in \mathcal{T}_m} \sum_{(j,m) \in i_t^+(m)} q_2(j,m) \left[\bar{S}(i,t) - \bar{S}(j,m)\right] &= 0. \end{aligned}$$

If the maximization problem characterizing the value is not uniquely solved by a node probability measure q, then the stopping time and the adjusted probability measure are also not uniquely defined. For instance, the value can be written as

$$V = \max_{\tau_1 \in \mathbb{T}} \max_{P_1 \in \mathcal{P}(\tau_1)} E^{p_1} G_{\tau_1} = \max_{\tau_2 \in \mathbb{T}} \max_{P_2 \in \mathcal{P}(\tau_2)} E^{p_2} G_{\tau_2}$$

with $X_{\tau_1}(i, t_1) = 1$, $X_{\tau_1}(j, t_2) = X_{\tau_1}(m, t_2) = 0$, $X_{\tau_2}(i, t_1) = 0$ and $X_{\tau_2}(j, t_2) = X_{\tau_2}(m, t_2) = 1$. Actually, the solution can be written with randomized stopping times. See the appendix C for an example, where (j, t_2) and (m, t_2) are terminal nodes.

As the stopping time is not unique, there are several *stopping time frontiers,* each one associated with a different stopping time. For any node inside all possible stopping time frontiers, the argument of the unique case solution applies and the agent does not have any incentive to exercise the American derivative. However, when the first stopping time frontier is reached, namely node (i, t_1), the American derivative may be exercised. At node (i, t_1) the value of the replicating portfolio, the payoff of the American derivative and the market value are the same. If the holder of the American derivative wants to guarantee the highest possible payoff at node (i, t_1), the derivative must be either exercised or sold at that node. If the holder of the American derivative wants to guarantee a given payoff at some successor of (i, t_1), there may be an incentive to exercise or sell the American derivative, and to use the proceeds to buy a replicating portfolio providing the same payoff as the American derivative at some successors, and a higher payoff at all other successors.

In the case where several stopping time frontiers coexist in this complete market setting, the exercise at any stopping time frontier before the *last* frontier provides a payoff equal to the value of the derivative. Also note that the last stopping time frontier is reduced to the role of a unique stopping time frontier, if the American derivative is not exercised at the previous frontiers.

7.2 Incomplete Markets

With incomplete markets the problem is more complex. In order to characterize the optimal exercise policy, we use the stopping time (τ_*^u) that solves the upper bound of the arbitrage free range of variation. Several points must be addressed.

First, if the reduced filtration is considered, the solution may involve randomized stopping times. Although it is not possible to conclude about an optimal exercise policy in this situation, we can still assign probabilities to the exercise of the American derivative at different nodes.

Second, if we consider the enlarged filtration, where ordinary stopping times are enough to describe the upper bound, the stopping time is not uniquely defined for all paths. In order to work out this case, we now extend the definition of a stopping time frontier for the case of incomplete markets as follows.

Definition 7.3 *A stopping time frontier is a pair* $\{(i,t), \mathbf{y}\}$ *such that* $X_{\tau^*}(i,t;\mathbf{y}) = 1$.

Remark 7.1 *Notice that complete markets corresponds to the case where there is only one vector* $\mathbf{y}$, *and the definition above reduces to the first definition of a stopping time frontier.*

Remark 7.2 *Note that for two different sets* $\mathbf{y}^1$ *and* $\mathbf{y}^2$ *with common values* $y_1^1 = y_1^2, y_2^1 = y_2^2, \ldots,$ *up to time* t, *if* $\tau(w;\mathbf{y}^1) = t$ *then* $\tau(w;\mathbf{y}^2) = t$. *Therefore* $X_{\tau^*}(i,t;\mathbf{y}^1) = 1 \Leftrightarrow X_{\tau^*}(i,t;\mathbf{y}^2) = 1$.

We define the interior of the stopping time frontier as follows.

Definition 7.4 *The interior of the stopping time frontier is the set of predecessors of the stopping time frontier.*

Even for price paths with a strictly positive q, the optimal stopping exercise is not uniquely defined using pure arbitrage arguments. For a given realization $\mathbf{y}$ of the stochastic process $\{y_t\}_{t\in\mathcal{T}}$, consider a path of the price process with a strictly positive q, and let (j,m) be the node such that $q(j,m) > 0$. Hence, $X_{\tau_*^u}(j,m;\mathbf{y}) = 1$ and, if node (j,m) is reached, the derivative will be exercised. However, it may be exercised at any predecessor of node (j,m). Let a predecessor of (j,m) be denoted by (k,n). The reason for the possibility of an American derivative be exercised at (k,n) is as follows. Using pure arbitrage arguments, it is possible to conclude that at any predecessor of (j,m), the price of the derivative is higher, or equal, to its payoff[3]. If the price is higher than the payoff at (k,n), *i.e.*, $P(k,n) > G(k,n)$, any rational agent is better off selling at (k,n), rather than exercising, the derivative. If the price of the derivative equals its payoff at (k,n),[4] *i.e.*,

$$P(k,n) = G(k,n), \tag{35}$$

a rational holder who wants to guarantee a given amount at (k,n) is indifferent between exercising or selling the derivative. In either case, the proceeds would suffice to buy a superreplicating portfolio that would assure the derivative value at (j,m). However, if this agent is concerned with the wealth at a successor of (k,n) different from (j,m), he may use the proceeds to buy a superreplicating portfolio providing the required payoff in that successor of (k,n).

However, if the American derivative is not exercised at any predecessor of the stopping time frontier, it will be exercised when the stopping time frontier is reached. The reason is that, at the frontier, the payoff is higher than the value of its replicating portfolio. However, if in a given path there is no node with a strictly positive q, the optimal stopping time can be such that $X_{\tau_*^u} = 1$ for some node with zero probability measure. As it is possible to have more than one node with zero probability measure, the exercise policy may not be uniquely defined.

A third and final point, is that the situation occurring in the complete market case leading to a non-unique solution of the dual problem, may also happen when markets are incomplete.

7.3 Complete *versus* Incomplete Markets

In this section we establish the following result.

[3] The reason is that $V^u(k,n) \geq G(k,n)$ and $V^l(k,n) \geq G(k,n)$. Hence, the arbitrage-free price must be higher than the payoff, i.e., $P(k,n) \geq G(k,n)$.

[4] In this case we should have $V^l(k,n) = G(k,n) \leq V^u(k,n)$

Proposition 7.3 *For every path such that the stopping time is unique, the stopping time frontier under complete markets is contained in the union of the stopping time frontier under incomplete markets and its interior.*

Proof. *Consider a given* $\mathbf{y}$*, such that* $y_t = 1$*. Let* $V_p^+(i,t)$ *be the value, at the node* (i,t)*, of the cheapest self-financing portfolio that, from time* $t+1$ *on, superreplicates all future payoffs of the American derivative. If the node* (i,t)*, belongs to the stopping time frontier, then* $G(i,t) > V_p^+(i,t)$*. Define* $V_1^+(i,t)$ *corresponding to* $V_p^+(i,t)$ *in the case of perfectly liquid markets. Both* $V_p^+(i,t)$ *and* $V_1^+(i,t)$ *are the solutions of minimization problems with the same objective function. Since the constraints characterizing* $V_1^+(i,t)$ *are contained in the set of constraints characterizing* $V_p^+(i,t)$*, it follows that* $V_1^+(i,t) \leq V_p^+(i,t)$*. Therefore, for any given* (i,t)*,* $G(i,t) < V_1^+(i,t) \Rightarrow G(i,t) < V_p^+(i,t;\mathbf{y})$*. Hence, nodes in the interior of the stopping time frontier under complete markets are also in the interior of the stopping time frontier under incomplete markets. On the other hand* $G(i,t) > V_p^+(i,t) \Rightarrow G(i,t) > V_1^+(i,t;\mathbf{y})$*. This means that nodes at the stopping time frontier under incomplete markets are not in the interior of the stopping time frontier under complete markets, completing the proof.* ■

We now turn to the case where there is not a unique stopping time. In that case, for each path w, pick the node $(k, t(w))$ such that $t(w) = \sup_s \left\{s : X_{\tau^*(w)}(i,s) = 1; \forall (i,s) \in w, \forall \tau^*(w)\right\}$. Let the set $\cup_{w \in \bar{\Omega}} (k, t(w))$ denote the envelope of the stopping time frontiers. We now have the following.

Proposition 7.4 *The envelope of the stopping time frontier under complete markets is contained in the union of the envelope of the stopping time frontiers under incomplete markets and its interior.*

Proof. Analogous to the proof above. ■

Corollary 7.5 *Under uniqueness of the stopping time, rational exercise of American options under incomplete markets may occur later than it would occur under complete markets.*

Remark 7.6 *This follows directly from the Proposition above. Notice, however, that American options may be exercised under incomplete markets before their optimal stopping time, under the condition specified by equation (35), i.e., under a situation of indifference. If not for that case, American options under incomplete markets will never be exercised before identical options under complete markets.*

8 Conclusion

We have shown that in the case of deterministic dry markets the bounds for the values of American derivatives are the supremum of the implied European derivatives, this supremum being taken over deterministic stopping times. In the probabilistic case there is an additional source of uncertainty, the existence or not of the market at given points in time, which can be interpreted as the realization of an additional stochastic process. If an enlarged filtration, resulting from the price process and the market existence process is considered, only ordinary stopping times are required to describe the upper and lower bounds. However, if the enlarged filtration were not considered, and the stopping times were defined using only the filtration induced by the price process, then they could be randomized, just as in Chalasani and Jha (2001).

In a complete market the arbitrage free value of the derivative is unique and equal to the value of the replicating portfolio. However, in our incomplete market framework, that no longer holds true. Ruling out arbitrage opportunities we simply obtain a range of variation for the value of the derivative. The arbitrage-free ranges of variation for the deterministic case, for the probabilistic case and when transactions are possible at all points in time are compared. We found out that range in the probabilistic case includes the range of the deterministic and the one if transactions are possible at all points in time. Moreover, the lower bound in the probabilistic case coincides with the one in the deterministic case. However, a relation cannot be established between the arbitrage-free range in the deterministic case and the one if transactions are possible at all points in time.

Moreover, when a complete market is considered the optimal exercise policy corresponds to the stopping time that is the supremum of the implied European derivatives. Consistently, with the absence of a unique arbitrage-free price, if American options are considered in this setting, the optimal exercise policy is also not well defined. The reason is that there are paths where the stopping time is not uniquely defined and, in addition, if the filtration induced by the price process is considered, randomized stopping time must be used. However, we were able to show that, under our incomplete market setting, an American derivative is exercised before it would be in a complete market, only under a situation of indifference. In other words, we have shown that market incompleteness may delay the optimal exercise of American derivatives.

There are several pricing alternatives in the literature to characterize the market value, or simply to restrict the arbitrage free range of variation. The different approaches[5] used with European options to choose a value in the arbitrage-free range, or to restrict that range, could be helpful with American derivatives. Notice that an important drawback of the bounds obtained in the probabilistic case is that these bounds do not depend on the probability of market existence. However, if a statistical arbitrage concept,

[5] For instance, equilibrium or utility based approach, as in Rubinstein (1976), Davis (1997) ; risk/reward criterion as in Bernardo and Ledoit (2000), Cochrane and Saá-Requejo (2000) and Bondarenko (2002); and considering the market price of risk associated with non traded state variables as in Heston (1993).

in the spirit of Bondarenko (2003), rather than pure arbitrage is used, the bounds could depend on the probability of the existence of the market. This is a further line of research to be pursued in the next essay with European derivatives.

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A Proof of theorems 2.1 and 2.2

Proof of theorem 2.1

Proof. Consider some nonsimple node-measure $q \in Q$. We will construct two new node probability measures a and c that belong to Q such that $q = \frac{1}{2}(a+c)$. Hence, as any nonsimple node-measure is not an extreme point we can conclude that an extreme point has to be a simple node-measure.

If q is a nonsimple node-measure then there is a node (i,t) such that $q(i,t) > 0$ and $\sum_{\substack{\tau>t \\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in \mathbf{i}_t^+(\tau)} q(j,\tau) > 0$. Consider such a node (i,t). Fix some strictly positive ε such that $\varepsilon < q(i,t)$ and for any $q(j,k) > 0$, with $k \in \mathcal{T}_m$, $k > t$ and $(j,k) \in i_t^+(k)$, we have $\varepsilon < q(j,k)$. Define a node measure a that is identical to q everywhere except that

$$a(i,t) = q(i,t) - \varepsilon$$

and, for any (j,k) such that $q(j,k) > 0$ with $k \in \mathcal{T}_m$, $k > t$ and $(j,k) \in i_t^+(k)$,

$$a(j,k) = q(j,k)\left(1 + \frac{\varepsilon}{\sum_{(j,\tau)\in \mathbf{i}_t^+(\tau)} \sum_{\substack{\tau>t \\ \tau\in\mathcal{T}_m}} q(j,\tau)}\right).$$

Note that the total amount by which q is increased on all the successors of (i,t) matches the amount by which q is decreased at (i,t)-that is, a is just a redistribution of q, and so is also a node-measure. The above statement also hold for the node-function c constructed as a but with $-\varepsilon$ instead of ε. It is easy to see that $q = \frac{1}{2}(a+c)$.

In order to conclude that $a \in Q$ we need to check that the following conditions hold

$$a(j,t) \geq 0, \tag{36}$$

$$\sum_{t\in\mathcal{T}_m} \sum_{(j,t)\in\mathcal{J}_t} a(j,t) = 1 \tag{37}$$

and

$$\sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} a\left(j,\tau\right)\bar{S}\left(k,t'\right) = \sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} a\left(j,\tau\right)\bar{S}\left(j,\tau\right). \tag{38}$$

The constraints of equations (37) and (36) are trivially respected. In what concerns the constraint in equation (38) only the relevant path is being analyzed. The constraint presented in equation (38) can be written as

$$\begin{aligned} & \sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} \left[q\left(j,\tau\right)+\sigma\left(j,\tau\right)\right]\bar{S}\left(k,t'\right) \\ = & \sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} \left[q\left(j,\tau\right)+\sigma\left(j,\tau\right)\right]\bar{S}\left(j,\tau\right) \end{aligned} \tag{39}$$

where $\sigma\left(j,\tau\right)$ defined as

$$\sigma\left(j,\tau\right)=\begin{cases} 0 & , \begin{array}{c}\left(j,\tau\right) \text{ such that } \tau<t \text{ or}\\ \left(j,\tau\right)\in i^{+}_{t}\left(\tau\right) \text{ such that } q\left(j,\tau\right)=0\end{array} \\ -\varepsilon & , \ \left(j,\tau\right)=\left(i,t\right) \\ \frac{\varepsilon q(j,\tau)}{\sum_{(j,\tau)\in\mathbf{i}^{+}_{t}(\tau)}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,m)} & , \ \text{otherwise} \end{cases}$$

As

$$\sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} q\left(j,\tau\right)\bar{S}\left(k,t'\right) = \sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} q\left(j,\tau\right)\bar{S}\left(j,\tau\right)$$

equation (39) can be written as

$$\sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} \sigma\left(j,\tau\right)\bar{S}\left(k,t'\right) = \sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} \sum_{(j,\tau)\in\mathbf{i}^{+}_{t'}(\tau)} \sigma\left(j,\tau\right)\bar{S}\left(j,\tau\right). \tag{40}$$

In order to check that equation (40) holds two alternative situations will be considered. The first one is $t'\leq t$. In this case, equation (40) can be written as

$$\begin{aligned} & \varepsilon\bar{S}\left(k,t'\right)+\frac{\varepsilon}{\sum_{j\in i^{+}(m)}\sum_{\substack{m>t\\ m\in\mathcal{T}_m}} q\left(j,m\right)}\sum_{j\in i(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q\left(j,\tau\right)\bar{S}\left(k,t'\right) \\ = & \varepsilon\bar{S}\left(i,t\right)+\frac{\varepsilon}{\sum_{j\in i^{+}(m)}\sum_{\substack{m>t\\ m\in\mathcal{T}_m}} q\left(j,m\right)}\sum_{j\in i(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q\left(j,\tau\right)\bar{S}\left(j,\tau\right). \end{aligned}$$

As both members are equal to zero the equality holds. The second case to be considered is $t' > t$. In this case, equation (40) can be written as

$$\begin{aligned} & \frac{\varepsilon}{\sum_{j\in i(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,\tau)} \sum_{j\in i(\boldsymbol{\tau})}\sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} q(j,\tau)\,\bar{S}(k,t') \\ = & \frac{\varepsilon}{\sum_{j\in i(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,\tau)} \sum_{j\in i(\boldsymbol{\tau})}\sum_{\substack{\tau>t'\\ \tau\in\mathcal{T}_m}} q(j,\tau)\,\bar{S}(j,\tau) \end{aligned}$$

As in the previous case, both members are equal to zero and the equality holds. ■

Proof of theorem 2.2

Proof. This theorem is corollary 5.5 of Chalasani and Jha (2001) with nodes corresponding to trading dates $t \in \mathcal{T}_m$. So, the proof of theorem 2.2 follows the one presented there. In order to proof that for a given measure strategy-pair (P, X_τ) the simple node measure $q(i,t) = P(i,t)\,X_\tau(i,t)$ is the unique simple node measure we only need to check that $q(i,t) \geq 0$ and

$$\begin{aligned} & \sum_{t\in\mathcal{T}_m}\sum_{(j,t)\in\mathcal{J}_t} q(j,t) \\ = & \sum_{t\in\mathcal{T}_m}\sum_{(j,t)\in\mathcal{J}_t} P(i,t)\,X_\tau(i,t) \\ = & \sum_{(i,T)\in\mathcal{J}_T} P(i,T) \sum_{(j,t)\in w\supset(i,T)} X_\tau(j,t) = 1. \end{aligned}$$

On the other hand, for any give node (i,t) such that

$$q(i,t) + \sum_{j\in i^+(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,\tau) > 0$$

the adjusted probability measure is uniquely defined and given by

$$P(i,t) = q(i,t) + \sum_{j\in i^+(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,\tau)\,.$$

For all other nodes the adjusted probability measure is defined forward-inductively in the following way. Consider any non-terminal node (i, t_1)

with $q(i,t_1) > 0$. Then, for a given immediate successor of (i,t_1), denoted (j,t_2), set $P(j,t_2) = P(i,t_1)$. For all other immediate successors of (i,t_1) set the adjusted probability equal to zero. If (j,t_2) is a nonterminal node then same process applies until a terminal node is reached.

In order to define the adapted process X_τ, let us denote by w the path that contains (i,t) and consider a node (j,t_2) that is an immediate predecessor of (i,t), *i.e.*, $(j,t_2) \in i_t^-$. Then, X_τ is defined as follows

$$X_\tau(i,t) = \begin{cases} \mathbb{I}_{\{(i,t):q(i,t)>0\}} & , \text{ if } \exists (j,m) \in w : q(j,m) > 0 \\ 0 & , \begin{array}{l} \text{if } \forall (j,m) \in w, q(j,m) = 0 \text{ and} \\ q(i,t) + \sum_{j \in i^+(\boldsymbol{\tau})} \sum_{\substack{\tau > t \\ \tau \in \mathcal{T}_m}} q(j,\tau) > 0 \end{array} \\ \mathbb{I}_{\{(j,t_2):\pi(j,t_2)>0\}} & , \text{ otherwise} \end{cases}$$

where $\pi(j,t_2) = q(j,t_2) + \sum_{i \in j^+(\boldsymbol{\tau})} \sum_{\substack{\tau > t_2 \\ \tau \in \mathcal{T}_m}} q(i,\tau)$.

The argument to proof the uniqueness of $X_\tau(i,t)$ and $P(i,t)$ when $q(i,t) + \sum_{j \in i^+(\boldsymbol{\tau})} \sum_{\substack{\tau > t \\ \tau \in \mathcal{T}_m}} q(j,\tau)$ is strictly positive is as follows. We first consider the case where $q(i,t) > 0$ and then the case where $q(i,t) = 0$. When $q(i,t)$ is strictly positive, in order to have $q(i,t) = p(i,t) X_\tau(i,t)$, $X_\tau(i,t)$ has to be equal to one. Moreover, the adjusted probability measure $p(i,t)$ has to be equal to $q(i,t)$. On the other hand, when $q(i,t) = 0$, in order to have $q(i,t) = p(i,t) X_\tau(i,t)$, either $X_\tau(i,t) = 0$ or $p(i,t) = 0$. However, as node (i,t) has a sucessor wtih strictly positive adjusted probability measure $p(i,t)$ -because $\sum_{j \in i^+(\boldsymbol{\tau})} \sum_{\substack{\tau > t \\ \tau \in \mathcal{T}_m}} q(j,\tau) > 0$ and $p(j,m) = q(j,m)$ when $q(j,m) > 0$ - the adjusted probability measure at this node, which is the sum of the adjusted probability on all successors, must be strictly positive. Hence, $X_\tau(i,t)$ has to be equal to zero. Additionally, as the adjusted probability at any given point is the sum of the adjusted probability on all successors the adjusted probability measure is uniquely defined in all nodes such that

$$q(i,t) + \sum_{j \in i^+(\boldsymbol{\tau})} \sum_{\substack{\tau > t \\ \tau \in \mathcal{T}_m}} q(j,\tau) > 0.$$

Note that, $X_{\tau}(i,t)$ is not only uniquely defined either when $q(i,t)>0$ or there exists a predecessor of (i,t) with strictly positive $q(i,t)$, i.e., $q(i,t)+\sum_{j\in i^{+}(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,\tau)>0$. It is also uniquely defined in the paths which contain a node with strictly positive q, although $q(i,t)+\sum_{j\in i^{+}(\boldsymbol{\tau})}\sum_{\substack{\tau>t\\ \tau\in\mathcal{T}_m}} q(j,\tau)$ may be zero. ■

B Proof of theorem 4.3

Proof of theorem 4.3

Proof. The proof that $\bar{P}(i,t)$ is a adjusted probability measure is straightforward. In what concerns the randomized stopping time, X, we must check that $X(i,t)\geq 0$ and condition (17) is satisfied.

For any given node $(j,t+1)\in i_t^{+}$ such $\alpha(j,t+1)\neq 0$ we have

$$\begin{aligned}
X(i,t) &= \frac{\alpha(i,t)}{\bar{P}(i,t)}-\frac{\alpha(i,t)-\sum_{\{\mathbf{y}:y_t=1\}}p(i,t;\mathbf{y})X_{\tau}(i,t;\mathbf{y})}{\bar{P}(i,t)}\\
&= \frac{\alpha(i,t)}{\bar{P}(i,t)}-\frac{\sum_{(j,t+1)\in i_t^{+}}\alpha(j,t+1)}{\bar{P}(i,t)}\\
&= \frac{\alpha(i,t)}{\bar{P}(i,t)}-\frac{\alpha(j,t+1)}{\bar{P}(j,t+1)}.
\end{aligned}$$

Consider a given path such that at the terminal node (i,T) we have $\alpha(i,T)>0$. In that case

$$\sum_{(i,t)\in w}X(i,t)=1.$$

Consider a given path such that at the node (j,s) we have $\alpha(i,t)\neq 0$ and $\alpha(m,t+1)=0$. In that case

$$\sum_{(k,r)\in w}X(k,r)=1-\frac{\alpha(i,t)}{\bar{P}(i,t)}+X(i,t)+X(m,t+1). \tag{41}$$

Let $(h,t+1)$ be a successor of (i,t). If $\alpha(h,t+1)\neq 0$ then $\bar{P}(m,t+1)=0$. Moreover, as $\bar{P}(i,t)\neq 0$ then $X(m,t+1)=\frac{\sum_{(i,t+1)\in i_t^{+}(t+1)}\alpha(i,t+1)}{\bar{P}(i,t)}$. As a result, equation (41) can be written as

$$\begin{aligned}
\sum\nolimits_{(k,r)\in w} X(k,r) &= 1-\frac{\alpha(i,t)}{\bar{P}(i,t)}+\frac{\sum_{\{\mathbf{y}:y_t=1\}} p(i,t;\mathbf{y})\, X_\tau(i,t;\mathbf{y})}{\bar{P}(i,t)} \\
&\quad +\frac{\sum_{(i,t+1)\in i_t^+(t+1)} \alpha(i,t+1)}{\bar{P}(i,t)} \\
&= 1.
\end{aligned}$$

However, if there are not a successor $(h,t+1)$ of (i,t) such that $\alpha(h,t+1)\neq 0$ then

$$\begin{aligned}
\sum\nolimits_{(k,r)\in w} X(k,r) &= 1-\frac{\alpha(i,t)}{\bar{P}(i,t)}+X(i,t)+X(m,t+1) \qquad (42)\\
&\quad +\sum\nolimits_{\substack{(k,r)\in w\\ r\geq t+1}} X(k,r). \qquad (43)
\end{aligned}$$

$\bar{P}(m,t+1)$ can take two possible values: 0 and $\bar{P}(i,t)$. Let us consider the two possibilities:

- $\bar{P}(m,t+1)=0$. This situation is the same as the one just described.

- $\bar{P}(m,t+1)=\bar{P}(i,t)$. As $\alpha(i,t)=\sum_{\{\mathbf{y}:y_t=1\}} p(i,t;\mathbf{y})\, X_\tau(i,t;\mathbf{y})$ then $-\frac{\alpha(i,t)}{\bar{P}(i,t)}+X(i,t)=0$. Moreover, $X(m,t+1)=0$. For any $(k,r)\in m_{t+1}^+(r)$ such that $\bar{P}(k,r)=\bar{P}(m,t+1)$ then $X(k,r)=0$. For a given $(k,r)\in m_{t+1}^+(r)$ such that $\bar{P}(k,r)=0$ and $\bar{P}(i,r-1)$ with $(k,r)\in i_{r-1}^+$ then $X(k,r)=\frac{\sum_{(k,r)\in i_t^+(r)}\alpha(k,r)}{\bar{P}(m,t+1)}=0$

As a result, equation (42) is verified and the proof is complete. ■

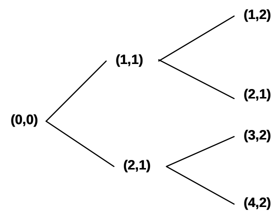
C Exercise Policy

Example C.1 *Consider a binomial tree with two periods. This framework is described in the following way where in parenthesis are identified the nodesThe primal problem can be written as*

$$V=\min_{\{\Delta(0,0),B(0,0),\Delta(1,1),B(1,1),\Delta(2,1),B(2,1)\}} \Delta(0,0)\,S(0,0)+B(0,0)$$

subject to the following conditions

$$\Delta(0,0)\,S(0,0)+B(0,0)\geq G(0,0),$$



$$\Delta(0,0)\,S(1,1)+B(0,0)\,R\geq G(1,1)\,,$$

$$\Delta(0,0)\,S(2,1)+B(0,0)\,R\geq G(2,1)\,,$$

$$\Delta(1,1)\,S(1,2)+B(1,1)\,R=G(1,2)\,,$$

$$\Delta(1,1)\,S(2,2)+B(1,1)\,R=G(2,2)\,,$$

$$\Delta(2,1)\,S(3,2)+B(2,1)\,R=G(3,2)\,,$$

$$\Delta(2,1)\,S(4,2)+B(2,1)\,R=G(4,2)\,,$$

and the self-financing constraints

$$\Delta(0,0)\,S(1,1)+B(0,0)\,R\geq\Delta(1,1)\,S(1,1)+B(1,1)$$

$$\Delta(0,0)\,S(2,1)+B(0,0)\,R\geq\Delta(2,1)\,S(2,1)+B(2,1)\,.$$

The dual problem can be written as

$$\max_{\substack{q(0,0)\\ \{q(i,1)\}_{i=1,2}\\ \{q(j,2)\}_{j=1,2,3,4}}} \quad q(0,0)\bar{G}(0,0)+\sum_{i=1}^{2}q(i,1)\,\bar{G}(i,t)+\sum_{i=1}^{4}q(i,2)\,\bar{G}(i,t)$$

such that

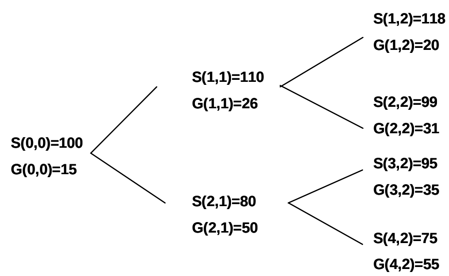
$$\sum_{i=1}^{2}q(i,2)\,S(1,1)=\sum_{i=1}^{2}q(i,2)\,\bar{S}(i,2)$$

$$\sum_{i=3}^{4}q(i,2)\,S(2,1)=\sum_{i=3}^{4}q(i,2)\,\bar{S}(i,2)$$

and

$$\left[\sum_{i=1}^{4} q\left(i,2\right)+\sum_{i=1}^{2} q\left(i,1\right)\right] S\left(0,0\right)=\sum_{i=1}^{4} q\left(i,2\right)\bar{S}\left(i,2\right)+\sum_{i=1}^{2} q\left(i,1\right)\bar{S}\left(i,1\right)$$

Consider the case where the interest rate is zero and the value of the underlying asset and the payoffs of the American derivative can be described asA



possible[6][7][8] *solution of the primal problem is*

	$\Delta\left(.,.\right)$	$B\left(.,.\right)$
$(0,0)$	-0.8	114
$(1,1)$	$-0,579$	$88,32$
$(2,1)$	-1	130

[6] The portfolio $\Delta(0,0)$ and $B(0,0)$ is uniquely defined because it is the only one that replicates at time $t=1$. At node node $(1,1)$ the value of the portfolio $[\Delta(0,0)$,$B(0,0)]$ is equal to $G(1,1)$ because the value of the portfolio that replicates $G(1,2)$ and $G(2,2)$ is smaller than $G(1,1)$. On the other hand, at node $(2,1)$ the value of the portfolio $[\Delta(0,0)$,$B(0,0)]$ is equal to $G(2,1)$ and to value of the replicating portfolio at $t=2$, because they coincide. Any other portfolio that superreplicates at time $t=1$ would be more expensive.

[7] The portolio $\Delta(1,1)$ and $B(1,1)$ is not uniquely defined because the value at node $(1,1)$ of the porfolio that replicates the payoffs of the American derivative at nodes $(1,2)$ and $(2,2)$ is smaller that $G(1,1)$.

[8] In node $(2,1)$ the value of the portfolio that replicates the payoff of the American derivative at nodes $(3,2)$ and $(4,2)$ is equal to the payoff of the American derivative, *i.e.*, $\Delta\left(2,1\right)S\left(2,1\right)+B\left(2,1\right)=G\left(2,1\right)$ Hence, any other portfolio that replicates the payoffs will have a cost higher than $G\left(2,1\right)$ resulting in a higher value of the function. As a result, the portfolio in node $(2,1)$ is unique.

Two possible solutions of the dual problem are given by

	$q^1(.,.)$	$q^2(.,.)$	$q^3(.,.)$
$(0,0)$	0	0	0
$(1,1)$	$0,6667$	$0,6667$	$0,6667$
$(2,1)$	$0,3333$	0	$0,2933$
$(1,2)$	0	0	0
$(2,2)$	0	0	0
$(3,2)$	0	$0,0833$	$0,01$
$(4,2)$	0	$0,25$	$0,03$

The correspondent stopping time and probability measures[9] *are given in the next table. The probability measure* $P \in \mathcal{P}$ *, that is uniquely defined, is presented in the last row.*

	$\tau^1(.,.)$	$P^1(.,.)$	$\tau^2(.,.)$	$P^2(.,.)$	$\tau^3_{\text{rand}}(.,.)$	$P^3(.,.)$	$P(.,.)$
$(0,0)$	0	$\mathbf{1}$	0	$\mathbf{1}$	0	$\mathbf{1}$	1
$(1,1)$	1	$\mathbf{\frac{2}{3}}$	1	$\mathbf{\frac{2}{3}}$	1	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{2}{3}}$
$(2,1)$	1	$\mathbf{\frac{1}{3}}$	0	$\mathbf{\frac{1}{3}}$	0.88	$\mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{3}}$
$(1,2)$	0	0	0	$\frac{2}{3}$	0	0	$\frac{22}{57}$
$(2,2)$	0	$\frac{2}{3}$	0	0	0	$\frac{2}{3}$	$\frac{16}{57}$
$(3,2)$	0	0	1	$\frac{1}{12}$	0.12	$\frac{1}{12}$	$\frac{1}{12}$
$(4,2)$	0	$\frac{1}{3}$	1	$\frac{1}{4}$	0.12	$\frac{1}{4}$	$\frac{1}{4}$

If ***assumption 1*** *holds the exercise policy can be given by the stopping time* τ^1 *or* τ^2. *However, if* ***assumption 2*** *holds the exercise policy is given by stopping time* τ^2.

[9] The probability measures are not uniquely defined. The probability in bold menas that it is uniquely defined in these nodes.