

## MSM ESTIMATORS OF EUROPEAN OPTIONS ON ASSETS WITH JUMPS

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This paper shows that, under some regularity conditions, the method of simulated moments estimator of European option pricing models developed by Bossaerts and Hillion (1993) can be extended to the case where the prices of the underlying asset follow Lévy processes, which allow for jumps, with no losses on their asymptotic properties, still allowing for the joint test of the model.

KEY WORDS: method of simulated moments, European options, simulated pricing, Lévy processes, asymptotic properties of estimators

### 1. INTRODUCTION

The main goal of this paper is to characterize method of simulated moments (MSM) estimators of European option pricing models on assets whose underlying price processes incorporate jumps.

Derivatives in general, and options in particular, are financial instruments defined as contracts whose payoffs depend on the stochastic evolution of the price of the underlying asset. If this payoff is to be received at a predefined date, termed *the maturity of the contract*, the instrument is said to be of the European type. To determine the value of these contracts was an old problem in the financial markets, until the 1973 seminal paper by Black and Scholes in which a closed formula for the price of European options in the case where the interest rates are constant and the price of the underlying asset follows a Geometric Brownian motion was established. Since then, most of the practice of pricing derivatives has been developed as perturbations of that result.

Recently, Bossaerts and Hillion (1993) characterized the properties of the MSM estimators in estimating and testing such models.

One first problem with this methodology is that it could not be directly extended to the important case of options incorporating the possibility of early exercise (the so-called American options). This has been done recently in Amaro de Matos (1995).

Another limitation of the methodology is the assumption used in both papers above concerning the continuity of the price processes of the underlying asset. In the language of options, this means that one must consider only underlying assets with continuous price processes that pay dividends at a continuous (possibly stochastic) rate.

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There are many known examples from real life where these assumptions are unrealistic. For example, dividends are usually paid discretely in time, not continuously. Also, options are frequently written on foreign currencies, and in that case the role of dividends is played by the difference between the domestic and foreign interest rates, as shown by Garman and Kohlhagen (1983). The stochastic process followed by foreign currency, however, is often subject to jumps, corresponding to revaluations or devaluations. In order to be applied to all these cases, the method must be extended to accommodate the impact of jumps.

Actually, this is not a recent concern. The empirical relevance of stochastic processes incorporating jumps in the valuation of derivatives has been recognized in the literature from the beginning Merton (1976). Much empirical work has been developed (see, among others, Akgiray and Booth 1988; Bodurtha and Courtadon 1987; Tucker and Pond 1988; and Tucker 1991) to justify the kind of modeling that has been made, but in the absence of a suitable methodology to simulate prices and jointly control simulation errors, most of this work lacks a sound econometric foundation.

This paper extends the MSM estimator of European option pricing models recently developed by Bossaerts and Hillion (1993) to the case where the underlying asset prices follow Lévy processes,<sup>1</sup> which allow for the possibility of jumps. In general terms, Lévy processes define a class of processes that include as particular cases Brownian motions and Poisson processes, providing prototypic examples for Markov processes as well as for semimartingales. Sufficient conditions are characterized to guarantee the asymptotic convergence and Normality of these MSM estimators in the context of such Lévy processes, still allowing for estimation and joint test of the model.

It is important to notice that the methodology here developed can be applied to modeling contexts that go much beyond the simple world of jump-diffusion processes. For example, interest has been centered recently<sup>2</sup> on the study of Variance-Gamma processes to describe the returns of financial assets. These are peaked, heavy-tailed processes that are empirically more suitable<sup>3</sup> for describing the stochastic nature of returns as compared to traditional diffusion processes such as the Black–Scholes model. Such processes can be seen as a limit of a particular sequence of compound Poisson processes in which the arrival rate of jumps diverges to infinity at the same time that the distribution of the size of those jumps concentrates its mass at zero. In that sense, such processes are very close to continuous time processes, although they are pure jump processes. These Variance-Gamma processes can be shown to have stationary, independent increments and to be continuous in Probability. They are, therefore, Lévy processes by definition, and the methodology of this paper can, therefore, be applied.

This paper is organized as follows. Section 2 presents the methodology, explaining how the simulation procedure is based on the pricing theory and introducing the method of moments to estimate and test the underlying model. Section 3 is devoted to the convergence properties of Euler schemes for simulation of stochastic processes driven by Lévy processes; Sections 4 and 5 use these properties to characterize the convergence of simulated prices and to describe the asymptotic properties of the MSM estimators. Section 6 provides some specific examples of processes for which the sufficient conditions for convergence and asymptotic Normality can be verified: Brownian motions, Poisson, and Variance-Gamma processes. The final section presents the conclusion.

<sup>1</sup> See Protter (1990) for definitions and a detailed description.

<sup>2</sup> See, for instance, the papers by Geman, Madan, and Yor (1998), Madan, Carr, and Chang (1998), Madan and Milne (1991), and Madan and Seneta (1990).

<sup>3</sup> See the arguments for this in Madan and Seneta (1990), for instance.

## 2. METHODOLOGY

In this section we first introduce the idea of pricing as the evaluation of expected discounted payoff to describe the nature of the Monte Carlo type of simulation to be used. Next, the panel data set methodology is introduced as in Bossaerts and Hillion (1993), in order to obtain a large enough sample size of simulated prices. Then the method of moments is presented as a suitable methodology to estimate the model and test the null hypothesis that the prices are formed in the market according to one such model.

### 2.1. Market Model and Option Prices

Assume an economy where there are  $d + 1$  marketed securities, the first of which, called *bond*, has a price  $s_0(\tau)$  and a positive payoff at any point in time. Suppose that the price  $s_0(\tau)$  satisfies

$$ds_0(\tau) = r(\tau)s_0(\tau)d\tau$$

where the function  $r$  is the short-term interest rate, and determines the *discount factor*

$$b(\tau) = \frac{1}{s_0(\tau)} = \exp \left\{ - \int_0^\tau r(t) dt \right\}.$$

The prices of the remaining  $d$  securities are assumed to follow a stochastic process, which possibly includes jumps. Let  $\theta$  be the vector of parameters characterizing such processes  $S(\tau, \theta)$ . The uncertainty about the time evolution of the prices is represented by a complete probability space  $(\Omega, \mathcal{F}, P)$  with an augmented filtration  $F = \{\mathcal{F}_\tau : \tau \in [0, \tau^*]\}$  of sub- $\sigma$ -algebras of  $\mathcal{F}$  satisfying the usual conditions.

As is well known,<sup>4</sup> in the absence of arbitrage opportunities, the price of any derivative such as an option can be written as its expected discounted payoff under a suitable risk-neutral probability measure  $P^*$ , equivalent to the measure  $P$ . From the primitives given above, it is possible to construct the process of the discounted payoff of the option at a known time  $\tau$  as

$$(2.1) \quad Q(\tau) = b(\tau)[S(\tau, \theta) - k]^+.$$

For a European option with maturity  $\tau^*$ , it may be shown (e.g., see Karatzas and Shreve 1988) that the smallest value of a hedging portfolio against the option (the value of the option) at any time  $t < \tau^*$  is given by the conditional expectation

$$(2.2) \quad V_e(t) = E^*[Q(\tau^*)|\mathcal{F}]_t,$$

where  $E^*[-]$  denotes expectation under the probability measure  $P^*$ .

### 2.2. Simulated Prices

We shall consider that the price process  $S$  follows a stochastic differential equation of the general form

$$S(\tau) = S(0) + \int_0^\tau f[S(t-)] dZ_t$$

where  $S(0)$  is an  $\mathbb{R}^d$ -valued random variable,  $S(t-) = \lim_{s \uparrow t} S(s)$ ,  $f$  is a  $d \times r$  matrix-valued function with argument in  $\mathbb{R}^d$ , and  $Z_t$  is an  $r$ -dimensional Lévy process, thus assuming the possibility of jumps, given the nature of such processes.

<sup>4</sup> See Duffie (1996) or Karatzas and Shreve (1988) for details.

For the sake of completeness, an elementary characterization of Lévy processes is made below (for further background on Lévy processes refer to Protter 1990). A Lévy process is an adapted process with stationary increments, independent of the past, and continuous in probability. One such process can be shown to have a unique modification which is càdlàg and is also a Lévy process. Letting  $Z_{t-} = \lim_{s \uparrow t} Z_s$ , we define the jump of the Lévy process at  $t$  as

$$\Delta Z_t = Z_t - Z_{t-}.$$

Now, let  $\Lambda$  be a Borel set in  $\mathbb{R}$ , bounded away from zero. The set function  $\Lambda \rightarrow N_t^\Lambda$ , defined as

$$N_t^\Lambda = \sum_{0 < s \leq t} 1_\Lambda(\Delta Z_s),$$

is simply a counting measure for each  $(t, \omega)$  that gives the number of times less than  $t$  such that the jumps of the Lévy process are in  $\Lambda$ . We let  $N_t(\omega, dx)$  denote the associated random measure such that

$$N_t^\Lambda = \int_\Lambda N_t(\cdot, dx).$$

It is also simple to verify that  $N^\Lambda$  is a Poisson process. Then

$$\nu(\Lambda) = E \left\{ \sum_{0 < s \leq 1} 1_\Lambda(\Delta Z_s) \right\}$$

is the parameter of the Poisson process  $N^\Lambda$  and is itself a measure, called the *Lévy measure* of the Lévy process  $Z$  with  $\int \min(1, x^2)\nu(dx) < \infty$ .

Let  $\|\cdot\|$  denote the usual Euclidean norm. The Lévy decomposition of  $Z$  is<sup>5</sup>

$$(2.3) \quad Z_t = \sigma W_t + \beta t + \int_{\|x\| < 1} x(N_t(\omega, dx) - t\nu(dx)) + \sum_{0 < s \leq t} \Delta Z_s 1_{\|\Delta Z_s\| \geq 1},$$

where  $W_t$  is a standard Brownian motion,  $\beta t$  is a drift term,  $N_t(\omega, dx)$  is the probability measure referred to above,  $\Delta Z_s = Z_s - Z_{s-}$  is the jump of  $Z$  at time  $s$ , and  $\nu$  is a probability measure on  $\mathbb{R}^d - \{0\}$  characterizing the Lévy process. A well-known example is the case where  $Z_t$  is a Lévy process with no drift, no Brownian term, and a finite Lévy measure, in which case it can be shown that  $Z_t$  is a compound Poisson process with jump arrival rate  $\lambda = \nu(\mathbb{R}^r)$  and jumps with distribution  $\frac{1}{\lambda}\nu$ .

In most cases it is impossible to evaluate the conditional probability distribution of the payoff  $Q$  given in expression (2.1) at the maturity date  $\tau^*$ . Then, as the value of the option cannot be directly evaluated by the expression (2.2) above, since the distribution of the underlying asset at maturity is not known, the price of the option must be simulated as in Bossaerts and Hillion (1993) by a Euler scheme. Typically, we propose to discretize the price of the underlying in time. Let  $\tau^*/M$  be the discretization step of the time interval  $[0, \tau^*]$  and let  $S^M(t)$  be the piecewise constant process defined by  $S^M(0) = S(0)$  and

$$S^M[(p+1)\tau^*/M] = S^M(p\tau^*/M) + f[S^M(p\tau^*/M)][Z_{(p+1)\tau^*/M} - Z_{p\tau^*/M}]$$

with  $p = 0, 1, 2, \dots, M - 1$ . The relevant issue here, to be developed in greater depth in Section 4, is to characterize the convergence of such simulated processes to the original ones and of certain classes of functionals of simulated asset prices to the original functionals of these asset prices, namely the prices of European options.

<sup>5</sup> See Theorem 4.2 in Protter (1990) and references therein.

The discretization of solutions of Stochastic Differential equations driven by Brownian motions and its properties have been discussed at length in the literature, as in Kloeden and Platten (1995). However, convergence properties of the discretized solution of SDE driven by Lévy processes have only recently been conveniently studied by Protter and Talay (1997). In their paper, suitable algorithmic procedures are presented for a simulation of the increments of a class of processes likely to appear in most applications.

From a practical point of view, the scheme presented here requires the ability to simulate the law of stationary and independent increments  $Z_{(p+1)\tau^*/M} - Z_{p\tau^*/M}$  in a computer. We can then obtain the value  $Q^M(\tau^*)$  for a sample path, as  $Q$  defined in (2.1) evaluated with the simulated process  $S^M(t)$ . The price of the option is the expectation taken over all possible paths. We can therefore simulate  $N$  different paths as in Bossaerts and Hillion (1993) and use stationarity and independence of the increments to approximate  $V_e$  by the expression

$$c^{[M, N]} = \frac{1}{N} \sum_{i=1}^N Q_i^M(\tau^*)$$

using the Law of Large Numbers when  $N \rightarrow \infty$  and the continuous-time limit when  $M \rightarrow \infty$ . In fact, for the sake of characterizing the convergence of  $c^{[M, N]}$  to  $V_e$ , the properties of both limits are to be well controlled. The first one is well known from the probabilistic literature. The second one, the continuous-time limit for the class of processes we are dealing with, is at the heart of the paper by Protter and Talay (1997). Our goal in this paper is to control both rates of convergence and derive the implications for estimation and testing of the underlying pricing model.

### 2.3. The Null Hypothesis

The construction of the price as an expected value of the discounted payoff can be seen as an implication of the absence of arbitrage. The value obtained in the simulation above depends on the type of process assumed for the underlying asset, that is to say, on the assumed function  $f$  in the model one is dealing with, and also on the values taken for  $\theta$ , the vector parameterizing the process or, in other words, parameterizing the function  $f$  itself. Therefore, the simulated price  $c^{[M, N]}$  is a function of  $\theta$  and should be written  $c^{[M, N]}(\theta)$ . Comparing the price  $c$  observed in the market place with the simulated price  $c^{[M, N]}(\theta)$  by the so-called pricing error  $c - c^{[M, N]}$  allows for testing of these *joint null hypotheses*, namely the pricing model, the absence of arbitrage, and the stochastic model for the underlying asset price induced by the chosen  $f$ , which includes the assumption that there exists a value  $\theta^*$  of  $\theta$  for which the pricing error is zero.

However, a simple comparison may not be fair in the sense that there is typically some noise in the formation of the price of options that is beyond the prescription of the model. If this is taken into account, the pricing error associated with a given model should thus be seen as a random variable. Under the model, this variable should have all its probability mass centered at zero. One such random variable has all its moments equal to zero. What shall be empirically tested is essentially a first-order approximation to this statement, namely that

$$E[c(t, k, \tau^*) - c^{[M, N]}(S(t), k, \tau^*; \theta^*)] = 0,$$

where the expectation is taken with respect to the probability measure of the pricing errors, jointly with the specification of the model given above.

Additional moment conditions may complement the null hypothesis. Let  $x(t)$  denote the vector of observable variables at time  $t$ , namely the market price and the characteristics of the option at time  $t$  such as the time to maturity, the exercise price, and the value of the interest rates.

As discussed above, the expected pricing error of an option should be zero independently of being out or in the money, of high or low interest rates, or even of its time to maturity. The price itself may depend on these factors, but not the pricing error. However, the early empirical literature in this area has identified deviations from theoretical prices typically correlated with such factors. Hence, in order to incorporate the absence of correlation of the pricing errors with such factors, a set of  $I$  instrumental variables  $y_q(t)$  is defined for  $q = 1, 2, \dots, I$ , which are assumed to be uncorrelated with the pricing error and to be observable at time  $t$ . Clearly all the  $y_q(t)$  are elements of the vector  $x(t)$ , as is the constant number 1. Based on the pricing error above, it is possible to define for  $q = 1, 2, \dots, I$  an  $I$ -dimensional vector with components

$$g_q(x(t); \theta) = \left[ c(t, k, \tau^*) - c^{[M, N]}(S(t), k, \tau^*; \theta) \right] y_q(t)$$

such that, under the null hypothesis to be tested, there exists a value  $\theta^*$  such that for  $q = 1, 2, \dots, I$

$$E[g_q(x(t); \theta^*)] = 0,$$

the so-called *moment conditions*.

#### 2.4. Panel Dataset

The expectations in the moment conditions above are taken over the probability distribution of the pricing errors. Such distribution being unknown in general, these expectations can still be evaluated in the case where the pricing errors are stationary and ergodic. In that case, for a large enough number of observations in time, the ergodic theorem states that the time average of the observed pricing errors converges to the expectation above.

To get as many observations as needed, the data set must be structured in a panel, as in the paper by Bossaerts and Hillion (1993). For each time  $t$ , a number  $J$  of different option prices are observed. Let each one of these be labeled by  $j = 1, 2, \dots, J$ . Each set of  $J$  observations is made for different points in time  $t = t_1, t_2, \dots, t_T$ . The number  $T$  of different points in time considered for observations will be termed the sample size. In this case, for each  $t$  a number  $J$  of moment conditions must be considered and the exercise price and time to maturity shall be distinguished among them by the introduction of the notation  $k_j(t)$  and  $\tau_j^*(t)$  respectively. The labeling of the moment conditions must therefore be changed if, for each  $j = 1, 2, \dots, J$ , one settles  $q = (j - 1)I + 1, (j - 1)I + 2, \dots, jI$  and the moments are rewritten as

$$g_q(x(t); \theta) = \left[ c(t, k_j(t), \tau_j^*(t)) - c^{[M, N]}(S(t), k_j(t), \tau_j^*(t); \theta) \right] y_q(t)$$

such that the null hypothesis to be tested is  $E[g_q(x(t); \theta^*)] = 0$ . But, as stressed before, most important here is that the errors are ergodic and stationary. This is not true in general, at least in the way the errors are constructed. However, as in Bossaerts and Hillion, one can divide each moment condition by its exercise price  $k_j(t)$  without loss of generality. Since options exchanges always reset exercise prices with reference to the

going price of the underlying asset when new contracts are introduced, the redefined pricing error should be ergodic and stationary. For that reason we shall deal further with the normalized pricing errors

$$g_q(x(t); \theta) = \left[ c(t, k_j(t), \tau_j^*(t)) / k_j(t) - c^{[M, N]}(S(t) / k_j(t), 1, \tau_j^*(t); \theta) \right] y_q(t).$$

Let  $g(x(t); \theta)$  denote the  $IJ$  dimensional vector of all these normalized pricing errors, and since it is now possible to take time averages to approximate the expectation, write the  $IJ$  sample moment conditions as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=t_1}^{t_T} g_q(x(t); \theta) = 0$$

for  $q = 1, 2, \dots, IJ$ . It is interesting to recall the dependence in some parameters in order to simplify the understanding of what is being done. Remember that to obtain  $c^{[M, N]}$  the time to maturity of an option is partitioned into  $M$  intervals of equal size. Also, because there are  $N$  different path simulations for each observation, there are  $N$  different simulated discounted payoffs per observation. Finally, notice that there are  $J$  observations per point in time.

## 2.5. Method of Moments

The MSM estimator of the parameters of the model is defined as the value  $\theta(T)$  such that the sample moment conditions are made as close as possible to zero, with respect to some metric. The criterion function of the MSM estimator is introduced as follows. Let  $D(T)$  be an  $IJ \times IJ$  symmetric positive definite matrix that converges in probability to  $D^*$  as  $T \rightarrow \infty$ . The MSM estimator  $\theta(T)$  is defined as the value of  $\theta$  that minimizes the quadratic form  $\xi$  of the sample moment conditions

$$\xi(T, N, \theta) = \left( \frac{1}{T} \sum_{t=t_1}^{t_T} g_q(x(t); \theta) \right)' D(T) \left( \frac{1}{T} \sum_{t=1}^T g_q(x(t); \theta) \right)$$

or still  $\theta(T) = \arg \min \xi(T, N, \theta)$ .

Notice that  $\xi(T, N, \theta) \geq 0$ , but under the null hypothesis it should be equal to zero. The idea of the method is to provide a statistical foundation to establish the level of a significant difference from zero to reject the null hypothesis, provided that one has chosen the best possible value of  $\theta$  for the moment conditions to hold.

The properties of these estimators have been characterized by Hansen (1982) in the case where there is no need to simulate the moments (i.e., the distribution at maturity would be known and the integral defining the moments could be evaluated). The introduction of simulation procedure adds noise to the estimation process. Its effect on the behavior of the estimators and on the power of the test has been studied by Pakes and Pollard (1989) and by McFadden (1989). This, however, is not enough for option pricing, since there is a kink in the payoff structure to be simulated in the moment conditions. This sort of problem is relevant because asymptotic Normality is obtained through a Taylor expansion of the simulated function around the parameters to be estimated. In the case of options the problem has been solved by Bossaerts and Hillion (1993) through a convex Taylor-like expansion, and asymptotic consistency and Normality have been proved for the case of limiting continuous sample paths for the underlying asset. We shall present a similar proof for the case of Lévy processes, which may include jumps, after studying the convergence properties of simulated Lévy processes in the next section.

### 3. CONVERGENCE OF SIMULATED PROCESSES WITH JUMPS

When  $Z$  is a Brownian motion and  $S^M(t)$  is the process corresponding to the Euler scheme defined above, it has been shown by Talay and Tubaro (1990) that for  $f$  smooth enough and a smooth  $g$  with polynomial growth, then  $Eg[S^M(t)]$  approximates  $Eg[S(t)]$  with an error of order  $O(M^{-1})$ . Subsequently, Bally and Talay (1995, 1996) have shown that the same result holds for any bounded and measurable  $g$ , using stochastic calculus of variation.

In a recent paper by Protter and Talay (1997) a similar result is derived for the class of processes being considered here. With suitable technical requirements they show that the error  $Eg[S^M(t)] - Eg[S(t)]$  may be expanded in powers of  $1/M$  if the Lévy measure of  $Z$  has finite moments of order high enough. Otherwise the rate of convergence is slower and its speed depends on the behavior of the tails of the Lévy measure. Their results are as follows.

For  $K > 0, m > 0$  and  $p \in \mathbb{N} - \{0\}$ , set

$$\begin{aligned} \rho_p(m) &= 1 + \|b\|^2 + \|\sigma\|^2 + \int_{-m}^m \|z\|^2 \nu(dz) \\ &\quad + \|b\|^p + \|\sigma\|^p + \left(\int_{-m}^m \|z\|^2 \nu(dz)\right)^{p/2} \\ &\quad + \int_{-m}^m \|z\|^p \nu(dz) \end{aligned}$$

and  $\eta_{K,p}(m) = \exp[K\rho_p(m)]$ . For  $m > 0$ , define  $h(m) = \nu\{z; \|z\| \geq m\}$ . Then the following results hold.

**THEOREM 3.1.** *Suppose that the functions  $f$  and  $g$  are both of class  $C^4$  and have all derivatives up to fourth order bounded, and  $S(0) \in L^4(\Omega)$ . Then, there exists a strictly increasing function  $K(\cdot)$  depending only on  $d, r$ , and the  $L^\infty$  norm of the partial derivatives of  $f$  and  $g$  up to order 4 such that for the considered discretization scheme and for any integer  $m$*

$$\left| Eg[S^M(t)] - Eg[S(t)] \right| \leq 4\|g\|_{L^\infty} \left[ 1 - \exp(-h(m)T) \right] + \frac{\eta_{K(T),8}(m)}{M}.$$

In this first result, the rate of convergence depends on the rate of increase to infinity of the functions  $h$  and  $\eta$ . In order to see how this approximation works, consider the following example. Let  $Z$  be a compound Poisson process with Lévy measure

$$\nu(dz) = 1_{\mathbb{R}_+}(z) \frac{1}{1+z^9} dz.$$

It follows that, as  $m$  tends to infinity,  $\rho_8(m)$  is of order  $\log(m)$  and  $h(m)$  is of order  $1/m^8$ . Therefore, for smooth enough functions  $f$  and  $g$ , Theorem 3.1 gives

$$\left| Eg[S^M(t)] - E[gS(t)] \right| \leq \frac{m^K}{M} + \frac{1}{m^8}.$$

Further restrictions allow for better bounds, as in Theorem 3.2.



**THEOREM 3.2.** *Suppose that the function  $f(\cdot)$  is of class  $\mathcal{C}^4$  and has all derivatives up to fourth order bounded; the function  $g(\cdot)$  is also of class  $\mathcal{C}^4$  and moreover  $|\partial_I g(s)| = \mathcal{O}(\|s\|^N)$  for  $|I| = 4$  and some  $N \geq 2$ ; the initial value of the process is  $S(0) \in L^{M^*}(\Omega)$ , where  $M^* := \max(2N, 8)$ . Furthermore, suppose that for  $2 \leq \gamma \leq M^*$*

$$\int_{\|z\| \geq 1} \|z\|^\gamma v(dz) < \infty.$$

*Then, there exists an increasing function  $K(\cdot)$  such that for the considered discretization scheme and for any  $M$*

$$\left| Eg[S^M(t)] - Eg[S(t)] \right| \leq \frac{\eta_{K(T), M^*}(\infty)}{M}.$$

A further refinement, imposing restrictions to the eighth derivative leads to the following theorem.

**THEOREM 3.3.** *Suppose that the function  $f(\cdot)$  is of class  $\mathcal{C}^8$  and has all derivatives up to eighth order bounded; the function  $g(\cdot)$  is also of class  $\mathcal{C}^8$  and moreover  $|\partial_I g(s)| = \mathcal{O}(\|s\|^{\bar{N}})$  for  $|I| = 8$  and some  $\bar{N} \geq 2$ ; the initial value of the process is  $S(0) \in L^{M^+}(\Omega)$ , where  $M^+ := \max(2N, 16)$ . Moreover, suppose that for  $2 \leq \gamma \leq M^+$*

$$\int_{\|z\| \geq 1} \|z\|^\gamma v(dz) < \infty.$$

*Then there exists a function  $C(\cdot)$  and an increasing function  $K(\cdot)$  such that, for the considered discretization scheme and for any  $M$ ,*

$$\left| Eg[S^M(t)] - Eg[S(t)] \right| \leq \frac{C(T)}{M} + R_T^M.$$

Notice that if the process has bounded jumps the integral condition is automatically satisfied, as will be shown in some of the examples in Section 6 below.

#### 4. CONVERGENCE OF SIMULATED PRICES

The goal of this section is to show that the price in equation (2.2), which may be written as  $E(c^{[\infty, 1]})$ , can be well approximated by the simulated price. When studying the convergence of the simulated price to the latter, the object which convergence we are interested in may then be written as

$$E(c^{[\infty, 1]} - c^{[M, 1]}).$$

The main result concerning the convergence is not simply that the above difference goes to zero, but a characterization of its rate of convergence to zero. It may be stated as the following theorem.

**THEOREM 4.1.** *If  $f$  satisfies the smoothness conditions of Theorem 3.2 and  $M$  grows as the sample size increases in such a way that  $\sqrt{T}/M(T) \rightarrow 0$  as  $T \rightarrow \infty$ , then in this limit*

$$\sqrt{T} \left( Ec^{[\infty, 1]} - Ec^{[M, 1]} \right) \rightarrow 0.$$

*Proof.* In order to prove this result, the argument under expectation is rewritten as

$$\begin{aligned} c^{[M, 1]} - c^{[\infty, 1]} &= b(\tau^*) \max[S^M(\bar{\tau}^M) - 1, 0] \\ &\quad - b(\tau^*) \max[S(\bar{\tau}) - 1, 0] \\ &= b(\tau^*) [\max[S^M(\tau^*) - 1, 0] - \max[S(\tau^*) - 1, 0]], \end{aligned}$$

where, without loss of generality,  $k$  is made equal to 1. Use of the elementary inequality

$$\max[\alpha - 1, 0] - \max[b - 1, 0] \leq |\alpha - b|$$

allows us to write

$$c^{[M, 1]} - c^{[\infty, 1]} \leq b(\tau^*) |S^M(\tau^*) - S(\tau^*)|.$$

Consider the assumptions of Theorem 3.2. Since  $S_0$  has a known value, the function  $g$  is the identity and hence all its derivatives are bounded. Thus, provided that  $f$  satisfies the smoothness conditions of at least Theorem 3.2, it follows that as  $M \rightarrow \infty$ ,  $S^M$  converges in  $L^1$  to  $S$  at a rate  $O(M^{-1})$ . By the assumption that  $\sqrt{T}/M(T) \rightarrow 0$ , the result follows.  $\square$

## 5. PROPERTIES OF THE ESTIMATORS

The main result of this paper can be summarized as follows. Consider an option written on an asset whose price follows a Lévy process. Consider also a discretization scheme such that the time partition becomes finer and finer at a sufficiently fast rate. Provided that this Lévy process satisfies certain regularity conditions, the simulated option price converges to the theoretical option price as the sample size increases at the same rate as if the underlying asset was a diffusion as studied in the paper by Bossaerts and Hillion (1993). Hence, the main results follow. The first is consistency of the MSM estimator.

**THEOREM 5.1.** *Suppose that the function  $f$  is of class  $C^4$ , has all derivatives up to fourth order bounded, and that*

$$\int_{\|x\| \geq 1} \|x\|^\gamma \nu(dx) < \infty \quad \text{for } 2 \leq \gamma \leq 8.$$

*Then, if  $\sqrt{T}/M(T) \rightarrow 0$  as  $T \rightarrow \infty$ , the method of moments estimator  $\theta(T)$  converges in probability to the true value  $\theta^*$  as  $T \rightarrow \infty$ .*

*Proof.* It suffices to show uniform convergence in probability to zero of

$$1/T \sum_{i=1}^T g^{[M(T), N]}(\theta) - Eg(\theta).$$

The procedures for this proof follow those in Bossaerts and Hillion (1993), using the basic statistical properties of Monte Carlo simulation, the Law of Large Numbers, and the unbiasedness of sample expectations.  $\square$

The second result is asymptotic Normality. In order to present the result, define  $\Phi(N)(\theta^*) = V^{pe}(\theta^*) + \frac{1}{N}V^{se}(\theta^*)$  with

$$V^{pe}(\theta^*) = E[g(x(t); \theta^*)g(x(t); \theta^*)']$$

and  $V^{se}(\theta^*) = E\left[c^{[\infty, 1]}(\theta^*)c^{[\infty, 1]}(\theta^*)'\right]$ . Also, denote by  $G$  the matrix of derivatives of the moment conditions with respect to the parameters and recall that  $D^*$  is a symmetric, positive definite matrix. Then the following theorem holds.

**THEOREM 5.2.** *Suppose that the function  $f$  is of class  $C^4$ , has all derivatives up to fourth order bounded, and that*

$$\int_{\|x\| \geq 1} \|x\|^\gamma \nu(dx) < \infty \quad \text{for } 2 \leq \gamma \leq 8.$$

*Then, if  $\sqrt{T}/M(T) \rightarrow 0$  as  $T \rightarrow \infty$ , the random vector  $\sqrt{T}(\theta(T) - \theta^*)$  converges weakly as  $T$  grows without bound to a normally distributed random vector with mean zero and variance-covariance*

$$\begin{aligned} & [EG'(\theta^*)D^*EG(\theta^*)]^{-1}EG'(\theta^*)D^*\Phi(N)(\theta^*)D^*EG(\theta^*) \\ & \times [EG'(\theta^*)D^*EG(\theta^*)]^{-1}. \end{aligned}$$

*Proof.* The proof rephrases that in Bossaerts and Hillion (1993). □

## 6. SOME IMPORTANT EXAMPLES

In this section some familiar examples of processes are analyzed and the necessary conditions on the price process for the above theorems to hold are verified.

Notice that the condition  $\sqrt{T}/M(T) \rightarrow 0$  as  $T \rightarrow \infty$  has nothing to do with the properties of the stochastic process for the underlying asset. It simply relates the speed at which the partition must get finer and finer as the sample size increases for the results to hold.

### 6.1. Brownian Motion

The simplest example one may think of for a price process is a Brownian motion. From the Lévy decomposition of a process  $Z$  given in (2.3), a Brownian motion for  $Z$  is obtained by simply taking the measure  $\nu$  as a degenerated distribution with mass centered at the origin,  $\nu(dx) = \delta(x)dx$ .

If this is the case, it then follows immediately that for all positive  $\gamma$

$$\int_{\|x\| \geq 1} \|x\|^\gamma \nu(dx) = 0.$$

In particular, the price process  $S$  follows a Brownian motion if the function  $f(\cdot)$  is a constant. At this point notice that  $S$  will follow a geometric Brownian motion if  $f(\cdot)$  is taken as the identity function.

In both cases, it is immediate to check that the regularity conditions on  $f(\cdot)$  are satisfied. Therefore, the asymptotic properties of the estimators described in the theorems of Section 5 hold.

Also notice that this particular choice for the measure  $\nu$  implies that for all  $m$ ,  $h(m) = 0$  and  $\rho_p(m)$  is a constant independent of  $m$  for given  $p$  and parameters  $\sigma$  and  $\beta$ . Therefore,  $\eta_{K(T), p}(m)$  is a constant as well. From the theorems in Section 3, one then finds that for both Brownian motion and Geometric Brownian motion,

$$|Eg[S^M(t)] - Eg[S(t)]| \leq \frac{\eta_{K(T), p}(m)}{M},$$

recovering the result of Talay and Tubaro (1990).

## 6.2. Poisson Process

A second example is a pure Poisson process. For  $Z_t$  to be a Poisson process, take  $\sigma = \beta = 0$  in the Lévy decomposition given by (2.3). Of course  $Z_t$  must have jumps of size 1, meaning that  $\nu(dx)$  is a degenerated distribution with mass centered at  $x = 1$ , or in other words,  $\nu(dx) = \delta(x - 1)dx$ .

With that choice of  $\nu(dx)$  it is clear that for all positive  $\gamma$

$$\int_{\|x\| \geq 1} \|x\|^\gamma \nu(dx) = 1,$$

which is finite. A suitable choice of the function  $f(\cdot)$  as the identity function makes it easy to check that the regularity conditions on  $f(\cdot)$  are satisfied. Therefore, the asymptotic properties of the estimators described in the above theorems of Section 5 hold.

Also notice that this particular choice for the measure  $\nu$  implies that for all  $m > 1$ ,  $h(m) = 0$  and  $\rho_p(m)$  is a constant independent of  $m$  for any given  $p$ . Therefore,  $\eta_{K(T), p}(m)$  is a constant as well. Then, for smooth enough  $g$ , Theorem 3.1 gives as before

$$|Eg[S^M(t)] - Eg[S(t)]| \leq \frac{\eta_{K(T), p}(m)}{M}.$$

A more general example is the case where  $Z_t$  is a Lévy process with no drift, no Brownian term, and a finite Lévy measure, in which case it can be shown (e.g., see Protter and Talay 1997) that  $Z_t$  is a compound Poisson process with jump arrival rate  $\lambda = \nu(\mathbb{R}^r)$  and jumps with distribution  $\frac{1}{\lambda}\nu$ .

Thus, as far as  $\nu$  has finite moments up to eighth order (a Normal distribution would clearly do), the results of the theorems of Section 5 will hold.

It is clear from the examples above that a mixed Poisson jump-Brownian motion process will satisfy the theorems of Section 5 with  $\beta \neq 0$ ,  $\sigma \neq 0$ , and  $\nu(dx)$  being a degenerated distribution with mass centered at  $x = 1$ .

### 6.3. Variance-Gamma Processes

As in the examples above, the validity of the asymptotic properties of the estimators as described in Section 5 depends basically on verifying that

$$\int_{\|x\| \geq 1} \|x\|^\gamma \nu(dx)$$

is finite for at least some values of  $\gamma$ . The Lévy measure for the Variance-Gamma process is given (see Madan et al. 1998 or Madan and Seneta 1990) by

$$\nu(dx) = \frac{\exp[-(|x|/\sigma)\sqrt{2/v}]}{v|x|} dx$$

for  $x \neq 0$  and  $\nu(\{0\}) = 0$ ,

where  $v, \sigma$  are parameters of the model. One must thus verify for this one-dimensional distribution the behavior of

$$\int_{|x| \geq 1} |x|^\gamma \frac{\exp[-(|x|/\sigma)\sqrt{2/v}]}{v|x|} dx$$

for  $\gamma \geq 2$ . A simple change of variables  $u = (x/\sigma)\sqrt{2/v}$  transforms this integral into

$$\sigma\sqrt{2/v} \int_{\sqrt{2/v}/\sigma}^{\infty} u^{\gamma-1} \exp[-u] du < \infty.$$

Finiteness of this integral is valid for all  $\gamma \geq 2$ . The usual considerations about the regularity conditions on the function  $f(\cdot)$  guarantee that the asymptotic properties of Section 5 are valid if this specific Lévy measure is used as suggested in the literature.

## 7. CONCLUSION

This paper started by referring to a paper by Bossaerts and Hillion (1993) whose main result is the asymptotic consistency and Normality of the MSM estimators of European option pricing model, where the price of the underlying asset follows a stochastic law with continuous sample paths. It has been shown here that this result may be extended to a class of processes, namely Lévy processes, that includes the presence of possible jumps.

Of course, this is true only under a certain number of regularity conditions on the price process of the underlying asset, which are well characterized. Namely, it is sufficient for the results to hold that the Lévy measure has finite moments of high enough order, as is the case in the compound Poisson process considered, or at least that the tails of the Lévy measure behave well enough as described above.

It has been shown that such conditions hold for the most common processes used in the literature, such as the mixed Poisson jump-Brownian motion process. Also, it has been shown that more complex Lévy measures such as the empirically more flexible Variance-Gamma process also satisfy these conditions.

Also interesting would be to check whether this result can be extended or not in the case of American options. This case includes a different kind of work because of the additional statistical noise that the prices of such options incorporate, driven by the possibility of early exercise. Work is being carried out along this line of research.

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